Chapter 4. Systems of ODEs. Phase plane. Qualitative methods

C.O.S. Sorzano

Biomedical Engineering

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Outline

1. Systems of ODEs. Phase plane. Qualitative methods
   - Systems of ODEs as models
   - Basic theory of systems of ODEs. Wronskian
   - Constant-coefficient systems. Phase plane method
   - Criteria for critical points. Stability
   - Qualitative methods for nonlinear systems
   - Nonhomogeneous linear systems of ODEs
References

Systems of ODEs. Phase plane. Qualitative methods

- Systems of ODEs as models
  - Basic theory of systems of ODEs. Wronskian
  - Constant-coefficient systems. Phase plane method
  - Criteria for critical points. Stability
  - Qualitative methods for nonlinear systems
  - Nonhomogeneous linear systems of ODEs
Mixing tanks

Solution:

\[ y'_1 = \text{inflow-outflow} = \frac{y_2}{100} \left[ \frac{lb}{gal} \right] 2 \left[ \frac{gal}{min} \right] \frac{y_1}{100} \left[ \frac{lb}{gal} \right] 2 \left[ \frac{gal}{min} \right] \]

\[ y'_2 = \text{inflow-outflow} = \frac{y_1}{100} \left[ \frac{lb}{gal} \right] 2 \left[ \frac{gal}{min} \right] \frac{y_2}{100} \left[ \frac{lb}{gal} \right] 2 \left[ \frac{gal}{min} \right] \]

\[ y'_1 = -0.02y_1 + 0.02y_2 \]
\[ y'_2 = 0.02y_1 - 0.02y_2 \]

\[ \begin{pmatrix} y'_1 \\ y'_2 \end{pmatrix} = \begin{pmatrix} -0.02 & 0.02 \\ 0.02 & -0.02 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \mathbf{y}' = \mathbf{A}\mathbf{y} \]
We try a solution of the form

\[ y = xe^{\lambda t} \]

\[ y' = \lambda xe^{\lambda t} \]

Now we substitute into the ODE

\[ y' = Ay \]

\[ \lambda xe^{\lambda t} = Ae^{\lambda t} \]

\[ \lambda x = Ax \]

That is \( y = xe^{\lambda t} \) can be a solution of the ODE if \( x \) is an eigenvector of the matrix \( A \) and \( \lambda \) its associated eigenvalue.

\[ A \Rightarrow \begin{cases} 
\lambda_1 = 0, x_1 = (1, 1)^T \\
\lambda_2 = -0.04, x_2 = (1, -1)^T 
\end{cases} \]
The general solution is
\[ y = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} \]

\[ y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-0.04t} \]

To determine the coefficients \( c_1 \) and \( c_2 \), we impose the initial conditions \( y_1(0) = 0, y_2(150) \)

\[ c_1 \begin{pmatrix} 0 \\ 150 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow c_1 = 75, c_2 = -75 \]

The solution to the problem is
\[ y = 75 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 75 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-0.04t} \Rightarrow \]

\[ \begin{cases} y_1 = 75 - 75e^{-0.04t} \\ y_2 = 75 + 75e^{-0.04t} \end{cases} \]
**Systems of ODEs as models**

**Mixing tanks (continued)**

\[
y = 75 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 75 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-0.04t} \Rightarrow \begin{cases} y_1 = 75 - 75e^{-0.04t} \\ y_2 = 75 + 75e^{-0.04t} \end{cases}
\]
Systems of ODEs as models

Electrical network

Solution:

\[ LL_1' + R_1(l_1 - l_2) = E \Rightarrow l_1' = -R_1 l_1 + R_1 l_2 + E \]

\[ R_1(l_2 - l_1) + \frac{1}{C} \int l_2 \, dt + R_2 l_2 = 0 \Rightarrow -R_1 l_1' + (R_1 + R_2) l_2' = -\frac{1}{C} l_2 \]

\[ L = 1 \text{ henry} \quad C = 0.25 \text{ farad} \]
\[ E = 12 \text{ volts} \]
\[ R_1 = 4 \text{ ohms} \]
\[ R_2 = 6 \text{ ohms} \]
Systems of ODEs as models

Electrical network (continued)

\[
\begin{align*}
I'_1 &= -R_1 I_1 + R_1 I_2 + E \\
-R_1 I'_1 + (R_1 + R_2) I'_2 &= -\frac{1}{C} I_2 \\
I'_1 &= -4I_1 + 4I_2 + 12 \\
-4I'_1 + 10I'_2 &= -4I_2
\end{align*}
\]

\[
\begin{pmatrix} I'_1 \\ I'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -4 & 10 \end{pmatrix} \begin{pmatrix} I'_1 \\ I'_2 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 0 \end{pmatrix}
\]

The eigenvalues and eigenvectors of the matrix \( A \) are

\[
A \Rightarrow \left\{ \begin{array}{l}
\lambda_1 = -2, \mathbf{x}_1 = (2, 1)^T \\
\lambda_2 = -0.8, \mathbf{x}_2 = (1, 0.8)^T
\end{array} \right.
\]
Electrical network (continued)

\[ A \Rightarrow \left\{ \begin{array}{l} \lambda_1 = -2, \mathbf{x}_1 = (2, 1)^T \\ \lambda_2 = -0.8, \mathbf{x}_2 = (1, 0.8)^T \end{array} \right. \]

The general solution of the H problem is

\[ \mathbf{I} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} \]

For a particular solution of the NH problem we try a constant vector \( \mathbf{I} = \mathbf{a} \)

\[ \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ -1.6 & 1.2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} 12 \\ 4.8 \end{pmatrix} \Rightarrow a_1 = 3, a_2 = 0 \]

So the general solution of the NH problem is

\[ \mathbf{I} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \]
Electrical network (continued)

\[ I = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \]

To determine the unknown coefficients we impose the initial condition \( I(0) = 0 \)

\[ 0 = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2 \cdot 0} + c_2 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8 \cdot 0} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \Rightarrow c_1 = -4, \ c_2 = 5 \]

The solution to the problem is

\[ I = -4 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-2t} + 5 \begin{pmatrix} 1 \\ 0.8 \end{pmatrix} e^{-0.8t} + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \]
Electrical network (continued)

\[ L = 1 \text{ henry}, \quad C = 0.25 \text{ farad} \]

\[ E = 12 \text{ volts} \]

\[ I_1, I_2, I_3 \]

\[ R_1 = 4 \text{ ohms}, \quad R_2 = 6 \text{ ohms} \]

\[ t = 0 \]

\[ I(t) \]

\[ I_1(t), I_2(t) \]

\[ I_1, I_2 \]

\[ 0, 1, 2, 3, 4, 5 \]

\[ 0, 0.5, 1, 1.5, 2, 3, 4, 5 \]
Conversion of an $n$-th order ODE to a system

**Conversion of an ODE**

An $n$-th order ODE

$$y^{(n)} = F(t, y, y', ..., y^{(n-1)})$$

can be converted to a system of $n$ first-order ODEs by setting $y_1 = y$ and

$$y'_1 = y_2$$
$$y'_2 = y_3$$
$$...$$
$$y'_{n-1} = y_n$$
$$y'_n = F(t, y_1, y_2, ..., y_n)$$
Conversion of an $n$-th order ODE to a system

**Example**

$$my'' + cy' + ky = 0$$

**Solution:**

$$y'' = -\frac{c}{m}y' - \frac{k}{m}y$$

Now we write it as

$$y_1' = y_2$$

$$y_2' = -\frac{k}{m}y_1 - \frac{c}{m}y_2$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Its characteristic polynomial is

$$\left| \begin{array}{cc} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{array} \right| = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$
Conversion of an $n$-th order ODE to a system

Example (continued)

\[ \lambda^2 + \frac{c}{m} \lambda + \frac{k}{m} = 0 \]

Let us now give values, $m = 1$, $c = 2$, $k = 0.75$

\[ \lambda^2 + 2\lambda + 0.75 = 0 \Rightarrow \lambda_1 = -0.5, \lambda_2 = -1.5 \]

With eigenvectors

\[ \mathbf{v}_1 = (2, -1)^T, \mathbf{v}_2 = (1, -1.5)^T \]

So, the general solution is

\[ \mathbf{y} = c_1 \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} 1 \\ -1.5 \end{pmatrix} e^{-1.5t} \]

The first component is

\[ y_1 = y = 2c_1 e^{-0.5t} + c_2 e^{-1.5t} \]

The second component is its derivative.
Exercises

From Kreyszig (10th ed.), Chapter 4, Section 1:

- 4.1.1
- 4.1.12
1 Systems of ODEs. Phase plane. Qualitative methods
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Basic theory

In general, an ODE system is of the form

\[ \begin{align*}
    y'_1 & = f_1(t, y_1, ..., y_n) \\
    y'_2 & = f_2(t, y_1, ..., y_n) \\
    & \quad \Rightarrow \quad y' = f(t, y) \\
    & \quad \vdots \\
    y'_n & = f_n(t, y_1, ..., y_n)
\end{align*} \]

An Initial Value Problem needs \( n \) initial conditions

\[ \begin{align*}
    y_1(t_0) & = K_1, y_2(t_0) = K_2, ..., y_n(t_0) = K_n \Rightarrow y(t_0) = K
\end{align*} \]

Existence and uniqueness

Let \( f_1, f_2, ..., f_n \) be continuous functions with continuous partial derivatives \( \frac{\partial f_1}{\partial y_1}, \frac{\partial f_1}{\partial y_2}, \ldots, \frac{\partial f_n}{\partial y_n} \) in some domain \( R \) of the \( ty_1y_2...y_n \)-space containing the point \( (t_0, K_1, ..., K_n) \). Then the ODE system has a solution on some interval \( t_0 - \alpha < t < t_0 + \alpha \) satisfying the initial conditions, and this solution is unique.
Basic theory

Linear systems

\[ y'_1 = a_{11}(t)y_1 + a_{12}(t)y_2 + \ldots + a_{1n}(t)y_n + g_1(t) \]
\[ y'_2 = a_{21}(t)y_1 + a_{22}(t)y_2 + \ldots + a_{2n}(t)y_n + g_2(t) \]
\[ \ldots \]
\[ y'_n = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \ldots + a_{nn}(t)y_n + g_n(t) \]

\[ \Rightarrow y' = Ay + g \]

with

\[ A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \]

If \( g = 0 \), the system is homogeneous.

Existence and uniqueness

Let the \( a_{ij} \)'s and \( g_i \)'s functions be continuous functions of \( t \) in an open interval \( I \) containing \( t_0 \), then there exists a solution satisfying the initial conditions, and this solution is unique.
Basic theory

Superposition principle

The linear combination of any two solutions, \( y_1 \) and \( y_2 \), of the H problem is also a solution of the H problem.

Proof:

\[
\begin{align*}
y &= c_1 y_1 + c_2 y_2 \\
y' &= c_1 y_1' + c_2 y_2' \\
&= c_1 (Ay_1) + c_2 (Ay_2) \\
&= A(c_1 y_1 + c_2 y_2) \\
&= Ay
\end{align*}
\]
General solution. Wronskian

If the $a_{ij}$'s functions are continuous, then the general solution of the H problem can be written as

$$y = c_1y_1 + c_2y_2 + ... + c_ny_n$$

where $y_1, y_2, ..., y_n$ constitute a basis or fundamental system of solutions, and there is no singular solution. We can write the $n$ basis functions as the columns of a matrix $Y$

$$Y = (y_1 \ y_2 \ ... \ y_n)$$

and write the general solution as

$$y = Yc$$

The Wronskian is the determinant of $Y$

$$W = |Y|$$
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Constant coefficients systems

We try a function of the form

\[ y = xe^{\lambda t} \]

\[ y' = \lambda xe^{\lambda t} \]

And substitute it in the ODE

\[ \lambda xe^{\lambda t} = Axe^{\lambda t} \]

\[ \lambda x = Ax \]

That is \( y \) is a solution if \( x \) is an eigenvector of \( A \). If \( A \) has \( n \) distinct eigenvalues, then the general solution is

\[ y = c_1 x_1 e^{\lambda_1 t} + ... + c_n x_n e^{\lambda_n t} \]
Constant coefficients systems

The Wronskian of the basis of solutions is

\[ W = \begin{vmatrix} x_1 e^{\lambda_1 t} & \cdots & x_n e^{\lambda_n t} \\ e^{\lambda_1 t + \cdots + \lambda_n t} & x_1 & \cdots & x_n \end{vmatrix} \]

The exponential term cannot be 0, and the determinant of the matrix of eigenvectors cannot be 0 because they are linearly independent vectors since they correspond to distinct eigenvalues. This proves that there is no singular solution if all eigenvalues are distinct.
Phase-plane trajectories

Example

\[ y' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} y \]

Solution:

\[ A \Rightarrow \begin{cases} 
\lambda_1 = -2, x_1 = (1, 1)^T \\
\lambda_2 = -4, x_2 = (1, -1)^T 
\end{cases} \]

\[ y = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} \]
Phase-plane trajectories

Example (continued)

\[ y = c_1 x_1 e^{\lambda_1 t} + c_2 x_2 e^{\lambda_2 t} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} \]
Phase-plane trajectories

Critical points

A critical point is a point at which $y' = 0$, they are also called equilibrium solutions. Let us analyze the system

$$y' = Ay$$

and the slope of trajectories in the phase plane at a given point $(y_1, y_2)$

$$\frac{dy_2}{dy_1} = \frac{y'_2}{y'_1} dt = \frac{y'_2}{y'_1}$$

At critical points, this ratio becomes undefined ($\frac{0}{0}$). There are five types of critical points: improper nodes, proper nodes, saddle points, centers, and spiral points.
An improper node is a critical point at which all trajectories, except two of them, have the same limiting direction of the tangent. The two exceptional directions also have a limiting direction of the tangent which, however, is different.
Phase-plane trajectories

Example: Proper node

\[ y' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} y \Rightarrow y = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t \]

A proper node is a critical point at which every trajectory has a definite limiting direction and for any given direction \( d \), there is a trajectory having \( d \) as its limiting direction.
Example: Saddle point

\[ y' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y \Rightarrow y = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} \]

A saddle point is a critical point at which there are two incoming trajectories, two outgoing trajectories, and all the other trajectories in a neighborhood of the critical point bypass it.
Example: Center

\[ y' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} y \Rightarrow y = c_1 \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} + c_2 \begin{pmatrix} 1 \\ -2i \end{pmatrix} e^{-2it} \]

A **center** is a critical point that is enclosed by infinitely many closed trajectories.
Phase-plane trajectories

Example: Spiral point

\[ \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} y \Rightarrow y = c_1 \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+i)t} + c_2 \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1-i)t} \]

A spiral point is a critical point about which trajectories spiral, approaching the critical point or going away from it, as \( t \to \infty \).
Phase-plane trajectories

Example: Degenerate node

\[ y' = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} y \]

The problem is that \( A \) is not diagonalizable because it has a double eigenvalue at \( \lambda = 3 \) but only one associated eigenvector \( x_1 = (1, -1)^T \). One of the solutions is of the form:

\[ y_1 = x_1 e^{\lambda_1 t} \]

For the second solution we look for solution of the type

\[ y_2 = tx_1 e^{\lambda_1 t} + ue^{\lambda_1 t} \]

with a constant \( u \) vector.

\[ y_2' = x_1 e^{\lambda_1 t} + t\lambda_1 x_1 e^{\lambda_1 t} + \lambda_1 ue^{\lambda_1 t} \]
Example: Degenerate node (continued)

We now substitute in the ODE

\[ y'_2 = Ay_2 \]

\[ x_1 e^{\lambda_1 t} + t\lambda_1 x_1 e^{\lambda_1 t} + \lambda_1 u e^{\lambda_1 t} = tAx_1 e^{\lambda_1 t} + Aue^{\lambda t} \]

\[ x_1 e^{\lambda_1 t} + t\lambda_1 x_1 e^{\lambda_1 t} + \lambda_1 u e^{\lambda_1 t} = t\lambda_1 x_1 e^{\lambda_1 t} + Aue^{\lambda_1 t} \]

\[ x_1 e^{\lambda_1 t} + \lambda_1 u e^{\lambda_1 t} = Aue^{\lambda_1 t} \]

\[ x_1 + \lambda_1 u = Au \]

\[ (A - \lambda_1 I)u = x_1 \Rightarrow u = (0, 1)^T \]

So the general solution is

\[ y = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \left( t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} \right) \]
Phase-plane trajectories

Example: Degenerate node (continued)

\[ y = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \left( t \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t} \right) \]
Degenerate solutions

When the matrix $A$ is not diagonalizable, then we may complete the fundamental system with solutions of the form

$$y_2 = (tx_1 + v_1) e^{\lambda_1 t}$$

$$y_3 = \left( \frac{1}{2} t^2 x_1 + tv_1 + v_2 \right) e^{\lambda_1 t}$$

$$y_4 = \left( \frac{1}{3} t^3 x_1 + \frac{1}{2} t^2 v_1 + tv_2 + v_3 \right) e^{\lambda_1 t}$$

...
System ODEs

\[
y' = \begin{pmatrix} y_2 \\ -\sin(y_1) \end{pmatrix}
\]

MATLAB

```matlab
f = @(t,y) [y(2);-sin(y(1))]
vectfield(f,-2:.5:8,-2.5:.25:2.5)
hold on
for y20=0:0.3:2.7
    [ts,ys] = ode45(f,[0,10],[0;y20]);
    plot(ys(:,1),ys(:,2))
end
hold off
```
Exercises

From Kreyszig (10th ed.), Chapter 4, Section 3:

- 4.3.6
- 4.3.7
- 4.3.18
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Criteria for critical points

Critical point classification

\[ y' = Ay = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} y \]

Let’s analyze the characteristic polynomial and eigenvalues of \( A \)

\[ \det\{A - \lambda I\} = \lambda^2 - \text{Tr}\{A\} \lambda + \det\{A\} = 0 \]

Let us define

\[ \Delta = (\text{Tr}\{A\} - 4(\det\{A\})^2) \]

The eigenvalues are

\[ \lambda_{1,2} = \frac{\text{Tr}\{A\} \pm \sqrt{\Delta}}{2} \]
Criteria for critical points

Critical point classification (continued)

\[ \lambda_{1,2} = \frac{\text{Tr}\{A\} \pm \sqrt{\Delta}}{2} \]

<table>
<thead>
<tr>
<th>Type</th>
<th>Tr{A} = \lambda_1 + \lambda_2</th>
<th>det{A} = \lambda_1 \lambda_2</th>
<th>\Delta = (\lambda_1 - \lambda_2)^2</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node</td>
<td>&gt; 0</td>
<td>≥ 0</td>
<td>Real, same sign</td>
<td></td>
</tr>
<tr>
<td>Saddle point</td>
<td>&lt; 0</td>
<td></td>
<td>Real, opposite signs</td>
<td></td>
</tr>
<tr>
<td>Center</td>
<td>= 0</td>
<td>&lt; 0</td>
<td>Pure imaginary</td>
<td></td>
</tr>
<tr>
<td>Spiral point</td>
<td>\neq 0</td>
<td>&lt; 0</td>
<td>Complex</td>
<td></td>
</tr>
</tbody>
</table>
Criteria for critical points

Stable critical point

A critical point $P_0$ is **stable** if all trajectories of the ODE that at some instant are close to $P_0$ remain close to $P_0$ at all future times; precisely: if for every disk $D_\epsilon$ of radius $\epsilon$ with center $P_0$ there is a disk $D_\delta$ of radius $\delta$ with center $P_0$ such that every trajectory of the ODE that has a point $P_1$ (corresponding to $t = t_1$, say) in $D_\delta$ has all its points corresponding to $t \geq t_1$ in $D_\epsilon$. If a critical point is not stable, it is **unstable**.

![Diagram of stable critical point](image)

**Fig. 90.** Stable critical point $P_0$ of (1) (The trajectory initiating at $P_1$ stays in the disk of radius $\epsilon$.)
Asymptotically stable critical point

A critical point $P_0$ is **asymptotically stable** (stable and attractive) if $P_0$ is stable and every trajectory that has a point in $D_δ$ approaches $P_0$ as $t \to \infty$.

**Fig. 91.** Stable and attractive critical point $P_0$ of (1)
Criteria for critical points

Critical point classification (continued)

\[ \det\{A - \lambda I\} = \lambda^2 - \text{Tr}\{A\} \lambda + \det\{A\} = \lambda^2 - p\lambda + q = 0 \]

\[ \lambda_{1,2} = \frac{\text{Tr}\{A\} \pm \sqrt{\Delta}}{2} \]

<table>
<thead>
<tr>
<th>Type</th>
<th>( p = \text{Tr}{A} = \frac{\lambda_1 + \lambda_2}{2} )</th>
<th>( q = \det{A} = \lambda_1 \lambda_2 )</th>
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<tbody>
<tr>
<td>Asymptotically stable</td>
<td>(&lt; 0)</td>
<td>(0)</td>
</tr>
<tr>
<td>Stable</td>
<td>(\leq 0)</td>
<td>(&gt; 0)</td>
</tr>
<tr>
<td>Unstable</td>
<td>(&gt; 0)</td>
<td>or (&lt; 0)</td>
</tr>
</tbody>
</table>
Criteria for critical points

Critical point classification (continued)

Fig. 92. Stability chart of the system (1) with $p, q, \Delta$ defined in (5). Stable and attractive: The second quadrant without the $q$-axis. Stability also on the positive $q$-axis (which corresponds to centers). Unstable: Dark blue region
## Criteria for critical points

### Critical point classification (continued)

<table>
<thead>
<tr>
<th><strong>eigenvalues</strong></th>
<th><strong>linear system</strong></th>
<th><strong>nonlinear system</strong></th>
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<tbody>
<tr>
<td><strong>real</strong></td>
<td></td>
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<td>proper or improper node</td>
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<td>both neg.</td>
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<td>proper or improper node</td>
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<td></td>
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<td>pos. and neg.</td>
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<td>saddle point</td>
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<td>real part pos.</td>
<td>spiral point</td>
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<td></td>
<td>real part neg.</td>
<td>spiral point</td>
</tr>
<tr>
<td></td>
<td>real part zero</td>
<td>center</td>
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</tbody>
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Criteria for critical points

Example

\[ my'' + cy' + ky = 0 \]

Solution:

\[ y'' = -\frac{k}{m}y - \frac{c}{m}y' \]

We convert it to a system ODE with

\[ y_1 = y, y_1' = y_2 \]

\[ y' = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} y \]

\[ \det(A - \lambda I) = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} \]

From where

\[ p = -\frac{c}{m}, q = \frac{k}{m}, \Delta = \left( \frac{c}{m} \right)^2 - 4\frac{k}{m} \]
Criteria for critical points

Example (continued)

\[ p = -\frac{c}{m}, \quad q = \frac{k}{m}, \quad \Delta = \left( \frac{c}{m} \right)^2 - 4 \frac{k}{m} \]

**No damping.** \( c = 0, \ p = 0, \ q > 0, \) a center.

**Underdamping.** \( c^2 < 4mk, \ p < 0, \ q > 0, \ \Delta < 0, \) a stable and attractive spiral point.

**Critical damping.** \( c^2 = 4mk, \ p < 0, \ q > 0, \ \Delta = 0, \) a stable and attractive node.

**Overdamping.** \( c^2 > 4mk, \ p < 0, \ q > 0, \ \Delta > 0, \) a stable and attractive node.
Exercises

From Kreyszig (10th ed.), Chapter 4, Section 4:

- 4.4.3
- 4.4.14
- 4.4.17
Systems of ODEs. Phase plane. Qualitative methods

- Systems of ODEs as models
- Basic theory of systems of ODEs. Wronskian
- Constant-coefficient systems. Phase plane method
- Criteria for critical points. Stability
- Qualitative methods for nonlinear systems
- Nonhomogeneous linear systems of ODEs
Autonomous nonlinear systems

Qualitative methods allow analyzing a system without actually solving it. For autonomous nonlinear systems

\[ y' = f(y) \]

with a critical point \( y_0 \) we may shift the origin so that the \( y_0 \) is centered

\[ \tilde{y} = y - y_0 \]

\[ \tilde{y}' = f(\tilde{y} + y_0) \]

\[ \tilde{y}' = \tilde{f}(\tilde{y}) \]

and study the local behaviour of the system ODE around \( 0 \) as we have already done. For doing so, we may need to linearize the ODE.
Autonomous nonlinear systems

Linearization of autonomous nonlinear systems

\[ \dot{\tilde{y}}' = \tilde{f}(\tilde{y}) \approx A\tilde{y} \]

where \( A \) is the Jacobian of the function \( f \) evaluated at the origin \( 0 \):

\[
A = \begin{vmatrix}
\frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} & \cdots & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_n} \\
\frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2} & \cdots & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \tilde{f}_n}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_n}{\partial \tilde{y}_2} & \cdots & \frac{\partial \tilde{f}_n}{\partial \tilde{y}_n}
\end{vmatrix}_{\tilde{y}=0}
\]

Theorem

If \( \tilde{f} \) has continuous components and continuous partial derivatives in a neighbourhood of the critical point \( 0 \) and \( \text{det}\{A\} \neq 0 \), then the kind and stability of the critical point of the nonlinear system ODE is the same as those of the linearized system. Exceptions occur if \( A \) has equal or pure imaginary eigenvalues, then the nonlinear problem may have the same kind of critical point as the linearized system or a spiral point.
Autonomous nonlinear systems

Example: Free undamped pendulum

Gravity compensates the acceleration of the bob

\[ mL\ddot{\theta} + mg \sin(\theta) = 0 \]

\[ \ddot{\theta} + k \sin(\theta) = 0 \quad k = \frac{g}{L} \]

To find the critical points we convert the equation into a system ODE

\[ y_1 = \theta \]

\[ y_2 = y_1' \]

\[ y_2' + k \sin(y_1) = 0 \]

Equivalently

\[ \mathbf{y}' = \begin{pmatrix} y_2 \\ -k \sin(y_1) \end{pmatrix} \]
Example (continued)

\[ \mathbf{y}' = \begin{pmatrix} y_2 \\ -k \sin(y_1) \end{pmatrix} \]

The critical points are at \( \mathbf{y} = (\pi n, 0)^T \ (n \in \mathbb{Z}) \). Let’s analyze the one at \((0, 0)\).
Let’s calculate the Jacobian of \( \mathbf{f} \) at \((0, 0)\)

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\
\frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2}
\end{pmatrix}
\bigg|_{y=0}
= \begin{pmatrix} 0 & 1 \\ -k \cos(y_1) & 0 \end{pmatrix}
\bigg|_{y=0}
= \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix}
\]

To classify this critical point we note that

\[
\text{Tr}\{A\} = 0 \quad \text{det}\{A\} = k > 0
\]

So we conclude that \( \mathbf{y} = \mathbf{0} \) is a center (always stable). The same happens to all points \((0, 2\pi n)\) since the sin function is periodic with period \(2\pi\).
Example (continued)

\[ \begin{align*} 
\mathbf{y}' &= \begin{pmatrix} y_2 \\ -k \sin(y_1) \end{pmatrix} 
\end{align*} \]

Let’s analyze the critical point at \((\pi, 0)\). We center the critical point by doing

\[ \tilde{\mathbf{y}} = \mathbf{y} - \begin{pmatrix} \pi \\ 0 \end{pmatrix} \]

The system ODE becomes

\[ \begin{align*} 
\tilde{\mathbf{y}}' &= \begin{pmatrix} \tilde{y}_2 \\ -k \sin(\tilde{y}_1 + \pi) \end{pmatrix} 
\end{align*} \]

Let’s calculate the Jacobian of \(\tilde{\mathbf{f}}\) at \((0, 0)\)

\[ A = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} \\ \frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2} \end{pmatrix} \bigg|_{\tilde{y}=0} = \begin{pmatrix} 0 & 1 \\ -k \cos(\tilde{y}_1 + \pi) & 0 \end{pmatrix} \bigg|_{y=0} = \begin{pmatrix} 0 & 1 \\ k & 0 \end{pmatrix} \]
To classify this critical point we note that

$$\text{Tr}\{A\} = 0 \quad \text{det}\{A\} = -k < 0$$

So we conclude that $y = (\pi, 0)^T$ is a saddle point (unstable). The same happens to all points $(0, \pi + 2\pi n)$ since the sin function is periodic with period $2\pi$. 

(b) Solution curves $y_2(y_1)$ of (4) in the phase plane.

Example 1 (C will be explained in Example 4.)
Autonomous nonlinear systems

Example: Damped pendulum

\[ \theta'' + c\theta' + k \sin(\theta) = 0 \]

To find the critical points we convert the equation into a system ODE

\[
\begin{align*}
y_1 &= \theta \\
y_2 &= y_1' \\
y_2' &= y_1 - k \sin(y_1) - cy_2
\end{align*}
\]

Equivalently

\[
y' = \begin{pmatrix} y_2 \\ -k \sin(y_1) - cy_2 \end{pmatrix}
\]
Example (continued)

\[
y' = \left( \begin{array}{c} y_2 \\ -k \sin(y_1) - cy_2 \end{array} \right)
\]

Critical points are at the same location as in the free undamped pendulum \( y = (\pi n, 0) \). Let’s study the critical point at \((0, 0)\).

\[
A = \left( \begin{array}{cc} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{array} \right) \bigg|_{y=0} = \left( \begin{array}{cc} 0 & 1 \\ -k \cos(y_1) & -c \end{array} \right) \bigg|_{y=0} = \left( \begin{array}{cc} 0 & 1 \\ -k & -c \end{array} \right)
\]

\[
\text{Tr}\{A\} = -c < 0 \quad \text{det}\{A\} = k > 0 \quad \Delta = -c + 4k^2
\]

If \( \Delta < 0 \), then we have a stable and attractive spiral point. If \( \Delta > 0 \), then it is a stable and attractive node.
Example (continued)

\[ \mathbf{y}' = \begin{pmatrix} y_2 \\ -k \sin(y_1) - cy_2 \end{pmatrix} \]

Let's analyze the critical point at \((\pi, 0)\). We center the critical point by doing

\[ \tilde{\mathbf{y}} = \mathbf{y} - \begin{pmatrix} \pi \\ 0 \end{pmatrix} \]

The system ODE becomes

\[ \tilde{\mathbf{y}}' = \begin{pmatrix} \tilde{y}_2 \\ -k \sin(\tilde{y}_1 + \pi) - cy_2 \end{pmatrix} \]

Let's calculate the Jacobian of \(\tilde{\mathbf{f}}\) at \((0, 0)\)

\[
A = \begin{pmatrix}
\frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} \\
\frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2}
\end{pmatrix}_{\tilde{y} = 0}
= \begin{pmatrix} 0 & 1 \\
-k \cos(\tilde{y}_1 + \pi) & -c \end{pmatrix}_{y = 0}
= \begin{pmatrix} 0 & 1 \\
k & -c \end{pmatrix}
\]
To classify this critical point we note that

\[ \text{Tr}\{A\} = -c \quad \text{det}\{A\} = -k < 0 \]

So we conclude that \( y = (\pi, 0)^T \) is a saddle point (unstable). The same happens to all points \( (0, \pi + 2\pi n) \) since the sin function is periodic with period \( 2\pi \).
Example: Lotka-Volterra population model

1. Rabbits have unlimited food supply. Hence, if there were no foxes, their number $y_1(t)$ would grow exponentially, $y_1' = ay_1$.

2. Actually, $y_1$ is decreased because of the kill by foxes, say, at a rate proportional to $y_1y_2$, where $y_2(t)$ is the number of foxes. Hence $y_1' = ay_1 - by_1y_2$, where $a > 0$ and $b > 0$.

3. If there were no rabbits, then $y_2(t)$ would exponentially decrease to zero, $y_2' = -ly_2$. However, $y_2$ is increased by a rate proportional to the number of encounters between predator and prey; together we have $y_2' = -ly_2 + ky_1y_2$, where $k > 0$ and $l > 0$.

Solution:

$$y_1' = ay_1 - by_1y_2$$

$$y_2' = ky_1y_2 - ly_2$$
Autonomous nonlinear systems

Example: Lotka-Volterra population model (continued)

Critical points are the solutions of:

\[ 0 = y_1' = y_1(a - by_2) \]
\[ 0 = y_2' = (ky_1 - l)y_2 \]

That is \((0, 0)\) or \((\frac{l}{k} \frac{a}{b})\). Let’s analyze \((0, 0)\)

\[
A = \left. \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{pmatrix} \right|_{y=0} = \left. \begin{pmatrix} a & -by_1 \\ ky_2 & -l \end{pmatrix} \right|_{y=0} = \begin{pmatrix} a & 0 \\ 0 & -l \end{pmatrix}
\]

Eigenvalues are \(\lambda_1 = a, \lambda_2 = -l\). They have different signs, so we have a saddle point.
Autonomous nonlinear systems

Example: Lotka-Volterra population model (continued)

For the critical point \( \left( \frac{l}{k}, \frac{a}{b} \right) \) we make the change of variables

\[
\tilde{y} = y - \left( \frac{l}{k} \frac{a}{b} \right)
\]

The system ODE becomes

\[
\tilde{y}' = \left( \tilde{y}_1 + \frac{l}{k} \right) \left( a - b \left( \tilde{y}_2 + \frac{a}{b} \right) \right) = \left( \tilde{y}_1 + \frac{l}{k} \right) \left( -b \tilde{y}_2 \right)
\]

Let’s calculate the Jacobian of \( \tilde{f} \) at \((0, 0)\)

\[
A = \left| \begin{array}{ll}
\frac{\partial \tilde{f}_1}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_1}{\partial \tilde{y}_2} \\
\frac{\partial \tilde{f}_2}{\partial \tilde{y}_1} & \frac{\partial \tilde{f}_2}{\partial \tilde{y}_2}
\end{array} \right|_{\tilde{y}=0} = \left| \begin{array}{ll}
-b\tilde{y}_2 & \left( \tilde{y}_1 + \frac{l}{k} \right) b \\
-\frac{k}{k} \tilde{y}_2 + \frac{a}{b} & k\tilde{y}_1
\end{array} \right|_{y=0} = \left( \begin{array}{cc}
0 & -\frac{l}{k} b \\
k\frac{a}{b} & 0
\end{array} \right)
\]
Example: Lotka-Volterra population model (continued)

\[ A = \begin{pmatrix} 0 & -\frac{I}{k}b \\ \frac{a}{b} & 0 \end{pmatrix} \]

We observe that

\[ \text{Tr}\{A\} = 0 \quad \text{det}\{A\} = al > 0 \]

So the critical point is a stable center. Let’s solve the equation around this critical point

\[ y_1' = -\frac{I}{k}b\tilde{y}_2 \]
\[ y_2' = k\frac{a}{b}\tilde{y}_1 \]

We rewrite the equation system as

\[ y_1' = -\frac{I}{k}b\tilde{y}_2 \]
\[ k\frac{a}{b}\tilde{y}_1 = y_2' \]
Example: Lotka-Volterra population model (continued)

\[ y_1' = -\frac{l}{k} b \tilde{y}_2 \]
\[ k \frac{a}{b} \tilde{y}_1 = y_2' \]

and multiply both equations

\[ k \frac{a}{b} \tilde{y}_1 y'_1 = -\frac{l}{k} b \tilde{y}_2 y'_2 \]

Integrating

\[ k \frac{a}{2b} \tilde{y}_1^2 = -\frac{l}{2k} b \tilde{y}_2^2 + C \]
\[ \frac{ak}{b} \tilde{y}_1^2 + \frac{bl}{k} \tilde{y}_2^2 = C \]
Example: Lotka-Volterra population model (continued)

\[
\frac{ak}{b} \ddot{y}_1 + \frac{bl}{k} \ddot{y}_2 = C \Rightarrow \frac{ak}{b} \left( y_1 - \frac{l}{k} \right)^2 + \frac{bl}{k} \left( y_2 - \frac{a}{b} \right)^2 = C
\]
Transformation to a first-order equation in the phase plane

Consider a second-order autonomous ODE

\[ F(y, y', y'') \]

We make the change of variables

\[ y_1 = y \]
\[ y_2 = y'_1 \]

And find \( y'' \) using the chain rule

\[ y'' = y'_2 = \frac{dy_2}{dt} = \frac{dy_2}{dy_1} \frac{dy_1}{dt} = \frac{dy_2}{dy_1} y_2 \]

The ODE becomes

\[ F \left( y_1, y_2, \frac{dy_2}{dy_1} y_2 \right) = 0 \]
Example: Free undamped pendulum

\[ \theta'' + k \sin(\theta) = 0 \]

Making the substitutions suggested by the method we get

\[ \frac{dy_2}{dy_1} y_2 + k \sin(y_1) = 0 \]
\[ y_2 dy_2 = -k \sin(y_1) dy_1 \]
\[ \frac{1}{2} y_2^2 = k \cos(y_1) + C \]
Exercises

From Kreyszig (10th ed.), Chapter 4, Section 5:

- 4.5.5
Systems of ODEs. Phase plane. Qualitative methods

- Systems of ODEs as models
- Basic theory of systems of ODEs. Wronskian
- Constant-coefficient systems. Phase plane method
- Criteria for critical points. Stability
- Qualitative methods for nonlinear systems

- Nonhomogeneous linear systems of ODEs
Nonhomogeneous linear systems of ODEs

\[ y' = A(t)y + g(t) \]

If the entries of the \( A \) matrix and \( g \) vector are continuous, then the general solution can be expressed as

\[ y = y_h + y_p \]

Method of undetermined coefficients

Valid for constant matrix \( A \) and \( g \) that is a sum of constant, powers, exponentials or sine/cosine functions.
Method of undetermined coefficients

Example

\[ y' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} y + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t} \]

Solution:
The general solution of the H problem is

\[ y_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} \]

Since the excitation signal \( e^{-2t} \) is also a solution of the H problem we try a particular solution of the form

\[ y_p = (tu + v)e^{-2t} \]

\[ y'_p = (-2tu + u - 2v)e^{-2t} \]
Example (continued)

Substituting in the ODE

\[ (-2t\mathbf{u} + \mathbf{u} - 2\mathbf{v})e^{-2t} = A(t\mathbf{u} + \mathbf{v})e^{-2t} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t} \]

\[ -2t\mathbf{u} + \mathbf{u} - 2\mathbf{v} = tA\mathbf{u} + A\mathbf{v} + \begin{pmatrix} -6 \\ 2 \end{pmatrix} \]

Identifying the coefficients of \( t \)

\[ -2\mathbf{u} = A\mathbf{u} \]

That is \( \mathbf{u} \) is an eigenvector of \( A \) associated to \( \lambda = -2 \)

\[ \mathbf{u} = a(1, 1)^T \]
Method of undetermined coefficients

Example (continued)

\[-2tu + u - 2v = tAu + Av + \begin{pmatrix} -6 \\ 2 \end{pmatrix}\]

Identifying the coefficients without \( t \)

\[u - 2v = Av + \begin{pmatrix} -6 \\ 2 \end{pmatrix}\]

\[(A + 2I)v = u - \begin{pmatrix} -6 \\ 2 \end{pmatrix}\]

We cannot solve as \((A + 2I)^{-1}(\ldots)\) because -2 is an eigenvalue of \( A \) and \( A + 2I \) is not invertible. Then

\[
\begin{pmatrix}
-3 & 1 \\
1 & -3
\end{pmatrix}
+ 
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}
= 
\begin{pmatrix} a \\ a \end{pmatrix}
- 
\begin{pmatrix} -6 \\ 2 \end{pmatrix}
\]
Method of undetermined coefficients

Example (continued)

\[
\left( \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} - \begin{pmatrix} -6 \\ 2 \end{pmatrix}
\]

\[
\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a + 6 \\ a - 2 \end{pmatrix}
\]

\[
\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a + 6 \\ 2a + 4 \end{pmatrix}
\]

For this system being compatible we need

\[2a + 4 = 0 \Rightarrow a = -2\]

Then

\[v_2 = v_1 + (-2 + 6) = v_1 + 4\]

We may simply take \( v_1 = 0 \)

\[v = \begin{pmatrix} 0 \\ 4 \end{pmatrix}\]
Finally

\[ \mathbf{y}_p = (t \begin{pmatrix} -2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix}) e^{-2t} \]

\[ \mathbf{y} = \mathbf{y}_h + \mathbf{y}_p = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} + \left( t \begin{pmatrix} -2 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \end{pmatrix} \right) e^{-2t} \]

\[ \mathbf{y} = \begin{pmatrix} c_1 - 2t \\ c_1 - 2t + 4 \end{pmatrix} e^{-2t} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} e^{-4t} \]
This is valid for non-constant $A$ and arbitrary $g$

$$y' = A(t)y + g(t)$$

If the general solution of the H problem is of the form

$$y_h = (y_1 \quad y_2 \quad \ldots \quad y_n) c = Y(t)c$$

Then we look for a solution of the form

$$y_p = Y(t)u(t)$$

$$y'_p = Y'u + Yu'$$

And substitute in the ODE

$$Y'u + Yu' = AYu + g$$
Method of variation of parameters

\[ Y'u + Y'u' = AYu + g \]

Since the columns of \( Y \) are solutions of the H problem we have

\[ Y' = AY \]

Then

\[ AYu + Y'u' = AYu + g \]

\[ Y'u' = g \]

\[ u' = Y^{-1}g \]
Method of variation of parameters

Example (same as for undetermined coefficients)

\[ y' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} y + \begin{pmatrix} -6 \\ 2 \end{pmatrix} e^{-2t} \]

Solution:
The general solution of the H problem is

\[ y_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = Y c \]

\[ Y^{-1} = \frac{1}{-2e^{-6t}} \begin{pmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{pmatrix} \]

\[ u' = Y^{-1} g = \frac{1}{2} \begin{pmatrix} e^{2t} & e^{2t} \\ e^{4t} & -e^{4t} \end{pmatrix} \begin{pmatrix} -6e^{-2t} \\ 2e^{-2t} \end{pmatrix} = \begin{pmatrix} -2 \\ -4e^{2t} \end{pmatrix} \]

\[ u = \int \begin{pmatrix} -2 \\ -4e^{2t} \end{pmatrix} dt = \begin{pmatrix} -2t \\ -2e^{2t} \end{pmatrix} \]
Method of variation of parameters

Example

\[ y_p = Y u = \begin{pmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{pmatrix} \begin{pmatrix} -2t \\ -2e^{2t} \end{pmatrix} = \begin{pmatrix} -2 - 2t \\ 2 - 2t \end{pmatrix} e^{-2t} \]

\[ y = y_h + y_p = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t} + \begin{pmatrix} -2 - 2t \\ 2 - 2t \end{pmatrix} e^{-2t} \]

\[ y = \begin{pmatrix} c_1 - 2 - 2t \\ c_1 + 2 - 2t \end{pmatrix} e^{-2t} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} e^{-4t} \]

We may compare to the previous solution

\[ y = \begin{pmatrix} c_1 - 2t \\ c_1 - 2t + 4 \end{pmatrix} e^{-2t} + \begin{pmatrix} c_2 \\ -c_2 \end{pmatrix} e^{-4t} \]
Exercises

From Kreyszig (10th ed.), Chapter 4, Section 6:

- 4.6.5
Outline

1. Systems of ODEs. Phase plane. Qualitative methods
   - Systems of ODEs as models
   - Basic theory of systems of ODEs. Wronskian
   - Constant-coefficient systems. Phase plane method
   - Criteria for critical points. Stability
   - Qualitative methods for nonlinear systems
   - Nonhomogeneous linear systems of ODEs