Problems

Author: First Year Biomedical Engineering

Supervisor: Carlos Oscar S. Sorzano

September 14, 2013
Chapter 2

Lay, 2.1.3

Let \( A = \begin{pmatrix} 2 & -5 \\ 3 & -2 \end{pmatrix} \). Calculate \( 3I_2 - A \) and \((3I_2)A\)

**Solution:**

\[
3I_2 - A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ -3 & 1 \end{pmatrix}
\]

\[
(3I_2)A = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 9 & -6 \\ 6 & -15 \end{pmatrix}
\]

Lay, 2.1.10

Let \( A = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix} \), \( B = \begin{pmatrix} -1 & 1 \\ 3 & 4 \end{pmatrix} \) and \( C = \begin{pmatrix} -3 & -5 \\ 2 & 1 \end{pmatrix} \). Verify that \( AB = AC \) and yet \( B \neq C \).

**Solution:**

\[
AB = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 21 & 21 \\ -7 & 7 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -3 & -5 \\ 2 & 1 \end{pmatrix} = AC
\]

Lay, 2.1.12

Let \( A = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix} \). Construct a \( 2 \times 2 \) matrix \( B \) such that \( AB \) is the zero matrix. Use two different nonzero columns for \( B \).

**Solution:** We search for a matrix \( B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \) such that

\[
AB = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 3b_{11} - 6b_{21} & 3b_{12} - 6b_{22} \\ 4b_{21} - 2b_{11} & 4b_{22} - 2b_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

This matrix equation gives us 4 equations

\[
3b_{11} - 6b_{21} = 0 \\
3b_{12} - 6b_{22} = 0 \\
4b_{21} - 2b_{11} = 0 \\
4b_{22} - 2b_{12} = 0
\]

The augmented matrix of this equation system is

\[
\begin{pmatrix} 3 & 0 & -6 & 0 & 0 \\ 0 & 3 & 0 & -6 & 0 \\ -2 & 0 & 4 & 0 & 0 \\ 0 & -2 & 0 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

Consequently, \( b_{11} = 2b_{21} \) and \( b_{12} = 2b_{22} \). That is, any matrix of the form

\[
B = \begin{pmatrix} 2b_{21} & 2b_{22} \\ b_{21} & b_{22} \end{pmatrix}
\]
yields $AB = 0$. One such example is $B = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$

**Lay, 2.1.18**

Suppose the third column of $B$ is all zeros. What can be said about the third column of $AB$?

**Solution:** Let us consider the different columns of $B$

\[
B = (b_1 \quad b_2 \quad b_3 \quad ...)
\]

The product of $AB$ is

\[
AB = A (b_1 \quad b_2 \quad b_3 \quad ...) = (Ab_1 \quad Ab_2 \quad Ab_3 \quad ...)
\]

If $b_3 = 0$, then

\[
Ab_3 = A0 = 0
\]

So, the third column is also 0.

**Lay, 2.1.19**

Suppose the third column of $B$ is the sum of the first two columns. What can be said about the third column of the product $AB$?

**Solution:** Let us consider the different columns of $B$

\[
B = (b_1 \quad b_2 \quad b_3 \quad ...)
\]

The product of $AB$ is

\[
AB = A (b_1 \quad b_2 \quad b_3 \quad ...) = (Ab_1 \quad Ab_2 \quad Ab_3 \quad ...)
\]

If $b_3 = b_1 + b_2$, then

\[
Ab_3 = A(b_1 + b_2) = Ab_1 + Ab_2
\]

That is, the third column of $AB$ is also the sum of the first and second columns of $AB$.

**Lay, 2.1.20**

Suppose that the first two columns, $b_1$ and $b_2$, of $B$ are equal. What can be said about the columns of $AB$? Why?

**Solution:** Let us consider the different columns of $B$

\[
B = (b_1 \quad b_2 \quad b_3 \quad ...)
\]

The product of $AB$ is

\[
AB = A (b_1 \quad b_2 \quad b_3 \quad ...) = (Ab_1 \quad Ab_2 \quad Ab_3 \quad ...)
\]

If $b_1 = b_2$, then

\[
Ab_1 = Ab_2
\]

So, both columns are also equal. Additionally, we may say that the columns of $AB$ are not linearly independent because there exists a linear combination of them that produces the vector 0.
Show that if the columns of $B$ are linearly independent, so are the columns of $AB$.

**Solution:** This statement is not true. For instance, the columns of

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

are linearly independent. However, given $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, the columns of

$$AB = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

are not linearly independent because the second column is twice the first one.

$AB$ is linearly independent if the columns of $A$ and $B$ are linearly independent.

**Lay, 2.1.23**

Let $A$ be an $m \times n$ matrix. Suppose there exists an $n \times m$ matrix $C$ such that $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $Ax = 0$ has only the trivial solution. Explain why $A$ cannot have more columns than rows.

**Solution:** If $x$ satisfies $Ax = 0$, then

$$CAx = C(Ax) = C0 = 0.$$  

But on the other side

$$CAx = (CA)x = I_n x = x.$$  

Consequently, $x = 0$. This shows that the equation $Ax = 0$ has no free variables. A requirement for this is that there are not more columns than rows.

**Lay, 2.1.24**

Suppose $A$ is a $n \times 3$ matrix whose columns span $\mathbb{R}^3$. Explain how to construct an $n \times 3$ matrix $D$ such that $AD = I_3$.

**Solution:** Let us define a generic matrix $D$

$$D = \begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & d_{m2} & \ldots & d_{mn} \end{pmatrix}$$

We need that $AD = I_3$. This gives us $9$ ($=3\cdot 3$) equations to find the matrix $D$. If the columns of $A$ span $\mathbb{R}^3$ and $n > 3$, the system is compatible indeterminate and there will be infinite solutions to the problem. If $n = 3$, there is a single solution to the problem. In the case that the columns of $A$ did not span $\mathbb{R}^3$, there would not be any solution to the problem.

**Lay, 2.1.25**
Suppose $A$ is an $m \times n$ matrix and there exist $n \times m$ matrices $C$ and $D$ such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$. [Hint: think of the product $CAD$.]

**Solution:** Let us compute

$$(CA)D = I_n D = D$$

On the other side, let us compute

$$C(AD) = CI_m = C$$

But we know that matrix multiplication is associative and, consequently, $C = D$.

By Exercise Lay 2.1.23 we know that $A$ cannot have more columns than rows, and by Exercise Lay 2.1.26 we know that $A$ cannot have more rows than columns. Consequently, the number of rows and columns must be the same and $m = n$.

**Lay, 2.1.26**

Let $A$ be an $m \times n$ matrix. Suppose there exists an $n \times m$ matrix $D$ such that $AD = I_m$ (the $m \times m$ identity matrix). Show that for any $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution. Explain why $A$ cannot have more rows than columns.

**Solution:** Let us consider the rows of $D = \begin{pmatrix} d_1 & d_2 & \ldots & d_m \end{pmatrix}$. The product $AD$ is

$$AD = \begin{pmatrix} Ad_1 & Ad_2 & \ldots & Ad_m \end{pmatrix} = I_m = \begin{pmatrix} e_1 & e_2 & \ldots & e_m \end{pmatrix}$$

where $e_i$ is the $i$-th column of $I_m$. For a particular column, we have

$$Ad_i = e_i$$

The columns of $I_m$ form a basis of $\mathbb{R}^m$. Therefore, for any $b \in \mathbb{R}^m$ it can be expressed as a linear combination of the $e_i$ vectors

$$b = \sum_{i=1}^m b_i e_i = \sum_{i=1}^m b_i Ad_i = A \left( \sum_{i=1}^m b_i d_i \right)$$

So we deduce, there exists a solution to the equation $Ax = b$ that is

$$x = \sum_{i=1}^m b_i d_i$$

If $A$ had more rows than columns, then it would not have a solution for every $b$ because there would be $b$’s for which the reduced echelon form has rows full of zeros and the independent terms are not $0$.

**Lay, 2.1.27**

Let $u = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix}$ and $v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$. Compute $u^T v$, $v^T u$, $uv^T$ and $vu^T$.

**Solution:**
\[
\mathbf{u}^T \mathbf{v} = \begin{pmatrix} -3 & 2 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = -3a + 2b - 5c
\]
\[
\mathbf{v}^T \mathbf{u} = \begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix} = -3a + 2b - 5c
\]
\[
\mathbf{u} \mathbf{v}^T = \begin{pmatrix} -3 \\ 2 \\ -5 \end{pmatrix} \begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} -3a - 3b - 3c \\ 2a + 2b + 2c \\ -5a - 5b - 5c \end{pmatrix}
\]
\[
\mathbf{v} \mathbf{u}^T = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} -3 & 2 & -5 \end{pmatrix} = \begin{pmatrix} -3a + 2b - 5c \\ -3b + 2b - 5b \\ -3c + 2c - 5c \end{pmatrix}
\]

**Lay, 2.2.7**

Let \( \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 5 & 12 \end{pmatrix} \), \( \mathbf{b}_1 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \), \( \mathbf{b}_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \), \( \mathbf{b}_3 = \begin{pmatrix} 2 \\ 6 \end{pmatrix} \), and \( \mathbf{b}_4 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \).

a. Find \( \mathbf{A}^{-1} \) and use it to solve the four equations \( \mathbf{Ax} = \mathbf{b}_1, \mathbf{Ax} = \mathbf{b}_2, \mathbf{Ax} = \mathbf{b}_3, \mathbf{Ax} = \mathbf{b}_4 \).

b. The four equations in part (a) can be solved by the same set of row operations, since the coefficients matrix is the same in each case. Solve the four equations in part (a) by reducing the augmented matrix \( \begin{pmatrix} \mathbf{A} & | & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{pmatrix} \).

**Solution:**

a. To find \( \mathbf{A}^{-1} \) we will apply row operations on the augmented matrix exploiting that \( (\mathbf{A} | \mathbf{I}) \sim (\mathbf{I} | \mathbf{A}^{-1}) \).

\[
\begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 5 & 12 & | & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & | & -\frac{6}{7} & -\frac{1}{7} \end{pmatrix}
\]

Now we use this inverse matrix to solve the linear equations

\[
\begin{pmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -9 \\ 4 \end{pmatrix}
\]
\[
\begin{pmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \end{pmatrix}
\]
\[
\begin{pmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}
\]
\[
\begin{pmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ -5 \end{pmatrix}
\]

b. Now, we will apply row operations on the augmented matrix suggested by the problem

\[
\begin{pmatrix} 1 & 2 & | & -1 & 1 & 2 & 3 \\ 5 & 12 & | & 3 & -5 & 6 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & | & -9 & 11 & 6 & 13 \\ 0 & 1 & | & -4 & -5 & -2 & -5 \end{pmatrix}
\]

5
Lay, 2.2.11

Let \( A \) be an invertible \( n \times n \) matrix, and let \( B \) be an \( n \times p \) matrix. Show that the equation \( AX = B \) has a unique solution \( X = A^{-1}B \).

**Solution:** Consider the columns of \( X \) and \( B \):

\[
X = \begin{pmatrix} x_1 & x_2 & \ldots & x_p \end{pmatrix} \\
B = \begin{pmatrix} b_1 & b_2 & \ldots & b_p \end{pmatrix}
\]

The matrix equation \( AX = B \) is a simultaneous set of equations:

\[
Ax_1 = b_1 \\
Ax_2 = b_2 \\
\vdots \\
Ax_p = b_p
\]

Since \( A \) is invertible, each equation has a unique solution given by

\[
x_1 = A^{-1}b_1 \\
x_2 = A^{-1}b_2 \\
\vdots \\
x_p = A^{-1}b_p
\]

Or what is the same

\[
X = \begin{pmatrix} A^{-1}b_1 & A^{-1}b_2 & \ldots & A^{-1}b_p \end{pmatrix} \\
= A^{-1} \begin{pmatrix} b_1 & b_2 & \ldots & b_p \end{pmatrix} \\
= A^{-1}B
\]

Lay, 2.2.13

Suppose \( AB = AC \), where \( B \) and \( C \) are \( n \times p \) matrices and \( A \) is invertible. Show that \( B = C \). Is this true, in general, if \( A \) is not invertible?

**Solution:** If \( A \) is invertible we multiply on the left by \( A^{-1} \) to obtain

\[
A^{-1}(AB) = A^{-1}(AC) \\
(A^{-1}A)B = (A^{-1}A)C \\
I_n B = I_n C \\
B = C
\]

If \( A \) is not invertible, then the statement is not generally true. For example, let

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

\[
AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = AC
\]

Lay, 2.2.17

Suppose \( A, B \) and \( C \) are invertible \( n \times n \) matrices. Show that \( ABC \) is also invertible by producing a matrix \( D \) such that \( (ABC)D = I = D(ABC) \)

**Solution:** The sought matrix \( D \) is
\[ D = C^{-1}B^{-1}A^{-1} \]

Let us check that this matrix is actually the inverse of \( ABC \).

\[
\begin{align*}
(ABC)D &= (ABC)(C^{-1}B^{-1}A^{-1}) = AB(CC^{-1})B^{-1}A^{-1} \\
D(ABC) &= (C^{-1}B^{-1}A^{-1})(ABC) = C^{-1}B^{-1}(A^{-1}A)BC \\
&= C^{-1}B^{-1}BC = C^{-1}C = I
\end{align*}
\]

**Lay, 2.2.19**

If \( A, B \) and \( C \) are invertible \( n \times n \) matrices, does the equation \( C^{-1}(A + X)B^{-1} = I_n \) have a solution, \( X \)? If so, find it.

**Solution:** If \( B \) and \( C \) are invertible, so are \( B^{-1} \) and \( C^{-1} \), and their inverses are \( B \) and \( C \), respectively. In this way, we may multiply on the left by \( C \) and on the right by \( B \) to obtain

\[
(BC^{-1} + X)B^{-1} = CI_n.
\]

In fact if \( ad - bc = 0 \), the matrix equation may have infinite solutions (if \( ab_2 - cb_1 = 0 \)) or no solution at all (if \( ab_2 - cb_1 \neq 0 \)). This implies that \( A \) is not invertible because if it were invertible for any \( b \in \mathbb{R}^2 \) the equation \( Ax = b \) would have a single solution.

**Lay, 2.2.21**

Explain why the columns of an \( n \times n \) matrix \( A \) are linearly independent when \( A \) is invertible.

**Solution:** If \( A \) is invertible we have shown (see Theorem 2.2, Chapter 3, Biomedical Engineering Notes) that for every \( b \in \mathbb{R}^n \), there is a unique solution of the equation \( Ax = b \). In particular, there exists a solution for the equation \( Ax = 0 \) that is \( x = A^{-1}0 = 0 \). Since the only solution of this problem is the trivial one, then by Theorem 6.1, Chapter 2, Biomedical Engineering Notes, the columns of \( A \) are linearly independent.

**Lay, 2.2.25**

Consider \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Show that if \( ad - bc = 0 \), then the equation \( Ax = b \) has more than one solution. Why does this imply that \( A \) is not invertible?

**Solution:** Let us reduce the augmented matrix \( (A|b) \).

\[
\begin{pmatrix}
a & b & b_1 \\
c & d & b_2
\end{pmatrix} \sim \begin{pmatrix}
a & b & 0 & ad - bc & ab_2 - cb_1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Let \( A = \begin{pmatrix} -25 & -9 & -27 \\ 536 & 185 & 537 \\ 154 & 52 & 143 \end{pmatrix} \). Find the second and third columns of \( A^{-1} \) without computing the first column.

**Solution:** Let us reduce the augmented matrix \( (A|e_1 \ e_2) \).
\[
\begin{pmatrix}
-25 & -9 & -27 & 0 & 0 \\
536 & 185 & 537 & 1 & 0 \\
154 & 52 & 143 & 0 & 1
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0.1126 & -0.1559 \\
0 & 1 & 0 & -0.5611 & 1.0077 \\
0 & 0 & 1 & 0.0828 & -0.1915
\end{pmatrix}
\]

The last two columns of the latter matrix are the two columns required by the problem.

**Lay, 2.3.13**

An \( m \times n \) upper triangular matrix is one whose entries below the main diagonal are 0's. When is a square upper triangular matrix invertible?

**Solution:** An upper triangular matrix is already in echelon form. It is row-equivalent to \( I_n \), and hence invertible, if its diagonal elements are different from 0. If any of the diagonal entries is zero, then there would be free variables in the equation system \( Ax = b \) and the matrix would not be invertible.

**Lay, 2.3.16**

If an \( n \times n \) matrix \( A \) is invertible, then the columns of \( A^T \) are linearly independent. Explain why.

**Solution:** By the Invertible Matrix Theorem, if \( A \) is invertible, so is \( A^T \). If \( A^T \) is invertible, then by the same theorem, the columns of \( A^T \) are linearly independent.

**Lay, 2.3.17**

Can a square matrix with two identical columns be invertible? Why or why not?

**Solution:** It cannot be invertible because the two columns are not linearly independent, and by the Invertible Matrix Theorem, if a matrix is invertible, then its columns are linearly independent.

**Lay, 2.3.33**

Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) given by \( T(x_1, x_2) = (-5x_1 + 9x_2, 4x_1 - 7x_2) \). Show that \( T \) is invertible and find a formula for \( T^{-1} \).

**Solution:** We may write the transformation as

\[
T(x_1, x_2) = \begin{pmatrix}
-5 & 9 \\
4 & -7
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

By defining the matrix \( A = \begin{pmatrix}
-5 & 9 \\
4 & -7
\end{pmatrix} \) and computing its inverse \( A^{-1} = \begin{pmatrix}
-0.0986 & 0.1268 \\
0.0563 & 0.0704
\end{pmatrix} \), we may write the inverse transformation as

\[
T(x_1, x_2) = \begin{pmatrix}
-0.0986 & 0.1268 \\
0.0563 & 0.0704
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

**Lay, 2.3.41**

Suppose an experiment leads to the following system of equations

\[
\begin{align*}
4.5x_1 + 3.1x_2 &= 19.249 \\
1.6x_1 + 1.1x_2 &= 6.843
\end{align*}
\]
a. Solve the previous equation system, and then, the equation system below in which the data on the right has been rounded to two decimal places.

\[
\begin{align*}
4.5x_1 + 3.1x_2 &= 19.25 \\
1.6x_1 + 1.1x_2 &= 6.84
\end{align*}
\]

b. The entries in the rounded system of equations differ from those of the exact system by less than 0.05%. Find the percentage error when using the solution of the rounded equation system as an approximation to the solution of the exact system.

**Solution:**

a. The solution of the exact equation system is

\[
x_{\text{exact}} = A^{-1} \begin{pmatrix} 19.249 \\ 6.843 \end{pmatrix} = \begin{pmatrix} 3.94 \\ 0.49 \end{pmatrix}
\]

The solution of the rounded equation system is

\[
x_{\text{rounded}} = A^{-1} \begin{pmatrix} 19.25 \\ 6.84 \end{pmatrix} = \begin{pmatrix} 2.90 \\ 2.00 \end{pmatrix}
\]

b. The error percentage is given for each variable as

\[
\epsilon_1 = 100 \frac{|x_{1,\text{exact}} - x_{1,\text{rounded}}|}{|x_{1,\text{exact}}|} = 100 \frac{|3.94 - 2.90|}{|3.94|} = 26.40\%
\]

\[
\epsilon_2 = 100 \frac{|x_{2,\text{exact}} - x_{2,\text{rounded}}|}{|x_{2,\text{exact}}|} = 100 \frac{|0.49 - 2.00|}{|0.49|} = 308.16\%
\]

**Lay, 2.4.15**

When a deep space probe is launched, corrections may be necessary to place the probe on a precisely calculated trajectory. Radio telemetry provides a stream of vectors, \(x_1, x_2, \ldots, x_k\), giving information at different times about how the probe’s position compares with its planned trajectory. Let \(X_k\) be the matrix \((x_1 \ x_2 \ \ldots \ x_k)\). The matrix \(G_k = X_kX_k^T\) is computed as the radar data is analyzed. Since the data vectors arrive at high speed, the computational burden could be severe. But partitioned matrix multiplication helps tremendously. Compute the column-row expansions of \(G_k\) and \(G_{k+1}\), and describe what must be computed in order to update \(G_k\) to form \(G_{k+1}\).

**Solution:** Let’s analyze first \(G_k\):

\[
G_k = X_kX_k^T = (x_1 \ x_2 \ \ldots \ x_k) \begin{pmatrix} x_1^T \\ x_2^T \\ \ldots \\ x_k^T \end{pmatrix} = \sum_{i=1}^{k} x_i x_i^T
\]

Similarly
Thus, it suffices to compute $x_{k+1}^T x_{k+1}^T$ and add it to the previous matrix $G_k$.

**Lay, 2.4.16**

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$. If $A_{11}$ is invertible, then the matrix $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the Schur complement of $A_{11}$. Likewise, if $A_{22}$ is invertible, the matrix $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is called the Schur complement of $A_{22}$. Suppose $A_{11}$ is invertible. Find $X$ and $Y$ such that

$$
\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} = 
\begin{pmatrix} A_{11}Y \\ XA_{11} + XA_{11}Y + S \end{pmatrix}
$$

Comparing this product to $A$ we derive the following equations:

$$
\begin{align*}
A_{11}Y &= A_{12} \\
XA_{11} &= A_{21} \\
XA_{11}Y + S &= A_{22}
\end{align*}
$$

That are solved like

$$
\begin{align*}
Y &= A_{11}^{-1}A_{12} \\
X &= A_{21}A_{11}^{-1}
\end{align*}
$$

We need to check that the last equation is verified

$$
\begin{align*}
XA_{11}Y + S &= A_{22} \\
(A_{21}A_{11}^{-1})A_{11}(A_{11}^{-1}A_{12}) + (A_{22} - A_{21}A_{11}^{-1}A_{12}) &= A_{22} \\
A_{21}A_{11}^{-1}A_{12} + A_{22} - A_{21}A_{11}^{-1}A_{12} &= A_{22} \\
A_{22} &= A_{22}
\end{align*}
$$

**Lay, 2.4.18**

Let $X$ be an $m \times n$ data matrix such that $X^TX$ is invertible, and let $M = I_m - X(X^TX)^{-1}X^T$. Add a column $x_0$ to the data to form $W = (X \ x_0)$. Compute $W^TW$. The $(1,1)$-entry is $X^TX$. Show that the Schur complement (Exercise Lay 2.4.16) of $X^TX$ can be written in the form $x_0^TMx_0$. It can be shown that $(x_0^TMx_0)^{-1}$ is the $(2,2)$-entry in $(W^TW)^{-1}$. This entry has a useful statistical interpretation under appropriate hypotheses.

**Solution:**

$$
W^TW = \begin{pmatrix} X^T \\ x_0^T \end{pmatrix} (X \ x_0) = \begin{pmatrix} X^TX & X^T x_0 \\ x_0^T X & x_0^T x_0 \end{pmatrix}
$$
The Schur complement is defined as $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$, that in this particular case is

$$S = x_0^T x_0 - x_0^T X (X^T X)^{-1} X^T x_0 = x_0^T (I_m - X (X^T X)^{-1} X^T) x_0 = x_0^T M x_0$$

Lay, 2.4.19

In the study of engineering control of physical systems, a standard set of differential equations is transformed by Laplace transforms into the following system of linear equations:

$$(A - sI_n) x + Bu = 0 \Rightarrow x = -(A - sI_n)^{-1} Bu$$

where $A$ is an $n \times n$ matrix, $B$ is an $n \times m$ matrix, $C$ is an $m \times n$ matrix, and $s$ is a variable. The vector $u \in \mathbb{R}^m$ is the “input” to the system, $y \in \mathbb{R}^m$ is the “output” of the system, and $x \in \mathbb{R}^n$ is the “state” vector. Actually, the vectors $u$, $x$ and $y$ are functions of $s$, but this does not affect the algebraic calculations of this exercise.

Assume $A - sI_n$ is invertible and view the previous equation as a system of two matrix equations. Solve the top equation for $x$ and substitute in the bottom equation. The result is an equation of the form $W(s)u = y$, where $W(s)$ is a matrix that depends on $s$. $W(s)$ is called the transfer function of the system because it transforms the input $u$ into the output $y$. Find $W(s)$ and describe how it is related to the partitioned system matrix of the equation above.

**Solution:** The first equation gives us

$$(A - sI_n)x + Bu = 0 \Rightarrow x = -(A - sI_n)^{-1} Bu$$

Now we go with the second equation and substitute this value into it

$$Cx + u = y$$

$$(C(-(A - sI_n)^{-1} Bu) + u) = y$$

$$(-C(A - sI_n)^{-1} B + I_m)u = y$$

$$(I_m - C(A - sI_n)^{-1} B)u = y$$

So, the transfer function is given by the matrix $W(s) = I_m - C(A - sI_n)^{-1} B$.

Lay, 2.5. Practice

Find an LU factorization of the matrix $A = \begin{pmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{pmatrix}$

**Solution:** We apply row operations on $A$ to reduce it to an upper triangular matrix and annotate the different matrices that we needed
\[
A = \begin{pmatrix}
2 & -4 & -2 & 3 \\
6 & -9 & -5 & 8 \\
2 & -7 & -3 & 9 \\
4 & -2 & -2 & -1 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
4 & -2 & -2 & -1 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
0 & 6 & 2 & -7 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
0 & 6 & 2 & -7 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
0 & 6 & 2 & -7 \\
\end{pmatrix}
\]

\[
E_1A = \begin{pmatrix}
2 & -7 & -3 & 9 \\
4 & -2 & -2 & -1 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
4 & -2 & -2 & -1 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
0 & 6 & 2 & -7 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
0 & 6 & 2 & -7 \\
\end{pmatrix}
\]

\[
E_1A = \begin{pmatrix}
2 \rightarrow r_2 - 3r_1 \\
6 \rightarrow 1 \quad 0 \quad 0 \\
2 \rightarrow 1 \quad 0 \\
4 \rightarrow 0 \\
-6 \rightarrow 0 \\
2 \rightarrow 0 \\
0 \rightarrow 0 \\
0 \rightarrow 0 \\
4 \rightarrow 0 \\
-6 \rightarrow 0 \\
2 \rightarrow 0 \\
0 \rightarrow 0 \\
0 \rightarrow 0 \\
0 \rightarrow 0 \\
0 \rightarrow 0 \\
0 \rightarrow 0 \\
\end{pmatrix}
\]

\[
E_2E_1A = \begin{pmatrix}
2 & -7 & -3 & 9 \\
4 & -2 & -2 & -1 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
4 & -2 & -2 & -1 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
0 & 6 & 2 & -7 \\
-6 & 3 & 3 & 4 \\
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & -3 & -1 & 6 \\
0 & 6 & 2 & -7 \\
\end{pmatrix}
\]

\[
E_2E_1A = \begin{pmatrix}
r_2 \rightarrow r_2 - 3r_1 \\
r_3 \rightarrow r_3 - r_1 \\
r_4 \rightarrow r_4 - 2r_1 \\
r_5 \rightarrow r_5 + 3r_1 \\
r_3 \rightarrow r_3 + r_2 \\
r_4 \rightarrow r_4 - 2r_2 \\
r_5 \rightarrow r_5 + 3r_2 \\
1 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 1 \\
0 \quad 0 \\
1 \quad 0 \\
0 \quad 1 \\
0 \quad 0 \\
1 \quad 0 \\
0 \quad 1 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \

This latter matrix is $U$ and $L$ is

$$
L = (E_9E_8E_7E_6E_5E_4E_3E_2E_1)^{-1} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-3 & 1 & 0 & 0 & 0 \\
-4 & 1 & 1 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 \\
2 & 1 & -2 & 0 & 1
\end{pmatrix}\quad L =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 \\
2 & 2 & -1 & 1 & 0 \\
-3 & -3 & 2 & 0 & 1
\end{pmatrix}
$$

Finally, we have

$$
A = LU \Rightarrow \begin{pmatrix}
2 & -4 & -2 & 3 \\
6 & -9 & -5 & 8 \\
4 & -2 & -2 & -1 \\
-6 & 3 & 3 & 4
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 \\
1 & -1 & 1 & 0 \\
2 & 2 & -1 & 1 \\
-3 & -3 & 2 & 0
\end{pmatrix} =
\begin{pmatrix}
2 & -4 & -2 & 3 \\
0 & 3 & 1 & -1 \\
0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

**Lay, 2.5.9**

Find an LU factorization of the matrix $A = \begin{pmatrix}
3 & 1 & 2 \\
-9 & 0 & -4 \\
9 & 9 & 14
\end{pmatrix}$

**Solution:** We apply row operations on $A$ to reduce it to an upper triangular matrix and annotate the different matrices that we needed

$$
A = \begin{pmatrix}
3 & 1 & 2 \\
-9 & 0 & -4 \\
9 & 9 & 14
\end{pmatrix} \quad r_2 \leftarrow r_2 + 3r_1 \quad E_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

$$
E_1A = \begin{pmatrix}
3 & 1 & 2 \\
0 & 3 & 2 \\
9 & 9 & 14
\end{pmatrix} \quad r_3 \leftarrow r_3 - 3r_1 \quad E_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-3 & 0 & 1
\end{pmatrix}
$$

$$
E_2E_1A = \begin{pmatrix}
3 & 1 & 2 \\
0 & 3 & 2 \\
0 & 6 & 8
\end{pmatrix} \quad r_3 \leftarrow r_3 - 2r_1 \quad E_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

This latter matrix is $U$ and $L$ is

$$
L = (E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1} =
\begin{pmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
3 & 0 & 1
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 2 & 1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
3 & 2 & 1
\end{pmatrix}
$$

Finally, we have
\[ A = LU \Rightarrow \begin{pmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix} \]

**Lay, 2.7.2**

Use matrix multiplication to find the image of the triangle with data matrix
\[ D = \begin{pmatrix} 4 & 2 & 5 \\ 0 & 2 & 3 \end{pmatrix} \] under the transformation that reflects a point through the y-axis. Sketch both the original triangle and its image.

**Solution:** The referred to transformation is the one whose matrix is \( A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

\[ D' = AD = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 5 \\ 0 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 5 \\ 0 & -2 & -3 \end{pmatrix} \]

**Lay, 2.7.3**

Find the \(3 \times 3\) matrix that translate by \((2, 1)\) and then rotate by \(90^\circ\) about the origin in 2D using homogeneous coordinates.

**Solution:** The required transformation is
\[ \tilde{A} = \begin{pmatrix} \cos(90^\circ) & \sin(90^\circ) & 0 \\ -\sin(90^\circ) & \cos(90^\circ) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \]

**Lay, 2.7.10**

Consider the following geometric 2D transformations: \(D\), a dilation (in which the \(x\) and \(y\) coordinates are scaled by the same factor); \(R\), a rotation; and \(T\), a translation. Does \(D\) commute with \(R\)? That is \((D(R(x))) = R(D(x))\) for all \(x \in \mathbb{R}^2\)? Does \(D\) commute with \(T\)? Does \(T\) commute with \(R\)?

**Solution:** The three proposed transformations can be written as matrix transformations in homogeneous coordinates
\[
D(x) = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{x}
\]
\[
R(x) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{x}
\]
\[
T(x) = \begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \tilde{x}
\]

Now we need to check whether \( D(R(x)) = R(D(x)) \)

\[
D(R(x)) = D \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}
\]
\[
= D \begin{pmatrix} r \cos(\alpha)x + \sin(\alpha)y \\ -r \sin(\alpha)x + r \cos(\alpha)y \\ 1 \end{pmatrix}
\]

On the other side

\[
R(D(x)) = R \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}
\]
\[
= R \begin{pmatrix} rx \\ ry \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} rx \\ ry \\ 1 \end{pmatrix}
\]
\[
= \begin{pmatrix} r \cos(\alpha)x + r \sin(\alpha)y \\ -r \sin(\alpha)x + r \cos(\alpha)y \\ 1 \end{pmatrix}
\]

So \( D(R(x)) = R(D(x)) \) and rotation commutes with dilation.

If we repeat the same exercise with dilations and translations

\[
D(T(x)) = D \begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}
\]
\[
= D \begin{pmatrix} x + \Delta x \\ y + \Delta y \\ 1 \end{pmatrix} = \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x + \Delta x \\ y + \Delta y \\ 1 \end{pmatrix}
\]
\[
= \begin{pmatrix} rx + r\Delta x \\ ry + r\Delta y \\ 1 \end{pmatrix}
\]

On the other side
\[
T(D(x)) = T \begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = T \begin{pmatrix} rx \\ ry \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} rx \\ ry \\ 1 \end{pmatrix} = \begin{pmatrix} rx + \Delta x \\ ry + \Delta y \\ 1 \end{pmatrix}
\]

So \( D(T(x)) \neq T(D(x)) \) and translation does not commute with dilation. Repeating once more the exercise with rotation and translation we would reach the conclusion that they do not commute.

**Lay, 2.7.12**

A rotation in \( \mathbb{R}^2 \) usually requires four multiplications. Compute the product below and show that the matrix for a rotation can be factored into three shear transformations (each of which requires only one multiplication).

\[
\begin{pmatrix}
1 - \tan(\frac{\phi}{2}) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
-\sin(\phi) & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 - \tan(\frac{\phi}{2}) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

**Solution:** Multiplying the three matrices we get

\[
\begin{pmatrix}
1 - \tan(\frac{\phi}{2}) & \tan(\frac{\phi}{2})(\tan(\frac{\phi}{2})\sin(\phi) - 2) & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

At this point, we make use of the trigonometric identity

\[
\tan(\frac{\phi}{2}) = \frac{\sin(\phi)}{1 + \cos(\phi)} = \frac{1 - \cos(\phi)}{\sin(\phi)}
\]

Then

\[
1 - \tan(\frac{\phi}{2})\sin(\phi) = 1 - \frac{\sin(\phi)}{1 + \cos(\phi)}\sin(\phi) = \frac{1 + \cos(\phi) - \sin^2(\phi)}{1 + \cos(\phi)} = \frac{\cos^2(\phi)}{1 + \cos(\phi)}
\]

Let us simplify now \( \tan(\frac{\phi}{2})(\tan(\frac{\phi}{2})\sin(\phi) - 2) \):

\[
\tan(\frac{\phi}{2})(\tan(\frac{\phi}{2})\sin(\phi) - 2) = \tan(\frac{\phi}{2})\left(\frac{1 - \cos(\phi)}{\sin(\phi)}\sin(\phi) - 2\right) = \tan(\frac{\phi}{2})(1 - \cos(\phi) - 2) = \tan(\frac{\phi}{2})(-1 - \cos(\phi)) = -\frac{\sin(\phi)}{1 + \cos(\phi)}(1 + \cos(\phi)) = -\sin(\phi)
\]

In summary

\[
\begin{pmatrix}
1 - \tan(\frac{\phi}{2}) & \tan(\frac{\phi}{2})(\tan(\frac{\phi}{2})\sin(\phi) - 2) & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
\cos(\phi) & -\sin(\phi) & 0 \\
\sin(\phi) & \cos(\phi) & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
That is, the multiplication of the three matrices above is the same as the application of a rotation matrix. But applying a rotation matrix involves 4 multiplications, while the application of the three matrices requires only 3.

Lay, 2.7.22

The signal broadcast by commercial television describes each color by a vector \((Y, I, Q)\). If the screen is black and white, only the \(Y\) coordinate is used (this gives a better monochrome picture than using CIE data for colors). The correspondence between \(YIQ\) and a “standard” \(RGB\) color is given by

\[
\begin{pmatrix}
Y \\
I \\
Q
\end{pmatrix} = \begin{pmatrix}
0.299 & 0.587 & 0.114 \\
0.596 & -0.275 & -0.321 \\
0.212 & -0.528 & 0.311
\end{pmatrix}
\begin{pmatrix}
R \\
G \\
B
\end{pmatrix}
\]

(A screen manufacturer would change the matrix entries to work for its \(RGB\) entries.) Find the equation that converts the \(YIQ\) data transmitted by the television station to the \(RGB\) data needed for the television screen.

**Solution:** If we consider the equation above to be

\[
\begin{pmatrix}
Y \\
I \\
Q
\end{pmatrix} = A
\begin{pmatrix}
R \\
G \\
B
\end{pmatrix},
\]

then

\[
\begin{pmatrix}
R \\
G \\
B
\end{pmatrix} = A^{-1}
\begin{pmatrix}
Y \\
I \\
Q
\end{pmatrix} = \begin{pmatrix}
1.0031 & 0.9548 & 0.6179 \\
0.9968 & -0.2707 & -0.6448 \\
1.0085 & -1.1105 & 1.6996
\end{pmatrix}
\begin{pmatrix}
Y \\
I \\
Q
\end{pmatrix}
\]

Lay, 2.8.1

Given the set \(H\) represented below (bold lines imply that those points belong to \(H\))

Give a specific reason of why the set is not a subspace of \(\mathbb{R}^2\)

**Solution:** For instance \(x = (1, 0)\) belongs to \(H\), but \(-x = (-1, 0)\) does not.

Lay, 2.8.2

Given the set \(H\) represented below (bold lines imply that those points belong to \(H\))
Give a specific reason of why the set is not a subspace of $\mathbb{R}^2$

**Solution:** For instance $x_1 = (-1, 1)$ and $x_2 = (2, 0)$ belong to $H$, but $x_1 + x_2 = (1, 1)$ does not.

*Lay, 2.8.5*

Let $v_1 = (1, 3, -4)$, $v_2 = (-2, -3, 7)$, and $w = (-3, -3, 10)$. Determine if $w$ is in the subspace of $\mathbb{R}^3$ generated by $v_1$ and $v_2$.

**Solution:** If $w$ is in the subspace generated by $v_1$ and $v_2$, then there must exist two constants $c_1$ and $c_2$ such that

$$w = c_1v_1 + c_2v_2$$

We may solve this problem through the augmented matrix

$$
\begin{pmatrix}
1 & -2 & -3 \\
3 & -3 & -3 \\
-4 & 7 & 10
\end{pmatrix}
\sim
\begin{pmatrix}
1 & -2 & -3 \\
0 & 3 & 6 \\
0 & 0 & 0
\end{pmatrix}
$$

The equation system is compatible determinate existing a single solution, and consequently, $w$ belongs to the subspace generated by $v_1$ and $v_2$.

*Lay, 2.9.1*

Given the basis $B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$ and $[x]_B = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Find $x$ and illustrate your answer.

**Solution:** Using the coordinates of $x$ in the basis $B$ we find

$$x = 3b_1 + 2b_2 = 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$$

The following figure illustrates this situation
La y, 2.9.3

\( x = \begin{pmatrix} 0 \\ 7 \end{pmatrix} \) is in a subspace \( H \) whose basis is \( B = \{b_1, b_2\} \) with \( b_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \)
and \( b_2 = \begin{pmatrix} -1 \\ 5 \end{pmatrix} \). Find the coordinates of \( x \) in the basis \( B \).

**Solution:** Let us look for the coordinates that satisfy

\[
x = c_1 b_1 + c_2 b_2
\]

For this, we will use the augmented matrix

\[
\begin{pmatrix}
2 & -1 & 0 \\
-3 & 5 & 7
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 2
\end{pmatrix}
\]

So, the coordinates of \( x \) in the basis \( B \) are \([x]_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\).

La y, 2.9.9

Consider \( A = \begin{pmatrix} 1 & 3 & 2 & -6 \\ 3 & 9 & 1 & 5 \\ 2 & 6 & -1 & 9 \\ 5 & 15 & 0 & 14 \end{pmatrix} \) and its echelon form

\[
\begin{pmatrix}
1 & 3 & 3 & 2 \\
0 & 0 & 5 & -7 \\
0 & 0 & 0 & 5 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Find bases for \( \text{Nul} \{A\} \) and \( \text{Col} \{A\} \).

**Solution:** The basis of \( \text{Nul} \{A\} \) is found by the equation system \( Ax = 0 \) whose augmented matrix is row-equivalent to

\[
\begin{pmatrix}
1 & 3 & 3 & 2 & 0 \\
0 & 0 & 5 & -7 & 0 \\
0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
We may calculate its reduced echelon form
\[
\begin{pmatrix}
1 & 3 & 3 & 2 & 0 \\
0 & 0 & 5 & -7 & 0 \\
0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \sim \begin{pmatrix}
1 & 3 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} = B
\]
This implies the following equations:
\[
\begin{align*}
x_1 &= -3x_2 \\
x_3 &= 0 \\
x_4 &= 0
\end{align*}
\]
So the basis of $\text{Nul}\{A\}$ is given by the non-pivot columns of $B$, i.e.,
\[
\text{Basis}\{\text{Nul}\{A\}\} = \{(-3, 1, 0, 0)\}
\]
The basis of $\text{Col}\{A\}$ is given by the pivot columns of $B$. The basis of the column space of $B$ is given by its first, third and fourth columns ($\{b_1, b_3, b_4\}$). Similarly, the basis of the column space of $A$ is given by its first, third and fourth columns ($\{a_1, a_3, a_4\}$), i.e.,
\[
\text{Basis}\{\text{Col}\{A\}\} = \{(1, 3, 2, 5), (2, 1, -1, 0), (-6, 5, 9, 14)\}
\]

**Lay, 2.9.19**

If the subspace of all solutions of $Ax = 0$ has a basis consisting of 3 vectors and if $A$ is a $5 \times 7$ matrix, what is the rank of $A$.

**Solution:** According to the rank theorem
\[
\text{Rank}\{A\} + \dim\{\text{Nul}\{A\}\} = n
\]
where $n$ is the number of columns of $A$. In this particular case,
\[
\text{Rank}\{A\} + 3 = 7 \Rightarrow \text{Rank}\{A\} = 4
\]

**Lay, 2.9.27**

Suppose vectors $b_1, b_2, ..., b_p$ span a subspace $W$, and let $\{a_1, a_2, ..., a_q\}$ by any set in $W$ containing more than $p$ vectors. Fill in the details of the following argument to show that $\{a_1, a_2, ..., a_q\}$ must be linearly dependent. First, let $B = (b_1 \ b_2 \ ... \ b_p)$ and $A = (a_1 \ a_2 \ ... \ a_q)$.

a. Explain why for each vector $a_j$, there exists a vector $c_j$ in $\mathbb{R}^p$ such that $a_j = Bc_j$.

b. Let $C = (c_1 \ c_2 \ ... \ c_q)$. Explain why there is a non-zero vector $u$ such that $Cu = 0$.

c. Use $B$ and $C$ to show that $Au = 0$. This shows that the columns of $A$ are linearly dependent.

**Solution:**
a. Each vector $a_j$ is in $W$, that is spanned by the $b_i$ vectors. That means that there exist some coefficients $c_{ji}$ such that

$$a_j = c_{j1}b_1 + c_{j2}b_2 + ... + c_{jp}b_p$$

or what is the same

$$a_j = Bc_j$$

b. Note that the $c_j$ vectors are in $\mathbb{R}^p$ since they have $p$ components. The problem stated that $q > p$, that is there are more $c_j$ vectors than $p$ (their dimension). By Theorem 6.2 of Chapter 2, we have that this set of equations is linearly dependent, that is, there exist some coefficients (not all of them zero) such that

$$u_1c_1 + u_2c_2 + ... + u_pc_p = 0$$

or

$$Cu = 0$$

c. Let us calculate $Au$. From point a, we know that $A = BC$, therefore

$$Au = (BC)u = B(Cu) = B0 = 0$$