Problems

Author:
First Year Biomedical Engineering

Supervisor:
Carlos Oscar S. Sorzano

September 22, 2013
1 Chapter 5

Lay, 5.1.1

Is $\lambda = 2$ an eigenvalue of \(
\begin{pmatrix}
3 & 2 \\
3 & 8 
\end{pmatrix}
\)? Why or why not?

**Solution:** To check whether $\lambda = 2$ is an eigenvalue or not, we test whether it is a solution of the equation

\[
\begin{vmatrix}
3 & 2 \\
3 & 8 
\end{vmatrix} - \lambda I = 0 \Rightarrow \begin{vmatrix}
3 & 2 \\
3 & 8 
\end{vmatrix} - 2 \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} = 0 \Rightarrow \begin{vmatrix}1 & 2 \\ 3 & 6\end{vmatrix} = 0 \Rightarrow 0 = 0
\]

Since we have got an identity ($0 = 0$), $\lambda = 2$ is a solution of the eigenvalue problem and it is an eigenvalue of the proposed matrix.

Lay, 5.1.3

Is \(\begin{pmatrix}1 \\ 3 \end{pmatrix}\) an eigenvector of \(\begin{pmatrix}1 & -1 \\ 6 & -4 \end{pmatrix}\)? If so, find the eigenvalue.

**Solution:** To check whether \(\begin{pmatrix}1 \\ 3 \end{pmatrix}\) is an eigenvector or not, we test whether it has the property

\[
\begin{pmatrix}1 & -1 \\ 6 & -4 \end{pmatrix} \begin{pmatrix}1 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix}1 \\ 3 \end{pmatrix}
\]

So, \(\begin{pmatrix}1 \\ 3 \end{pmatrix}\) is an eigenvector and its associated eigenvalue is $-2$.

Lay, 5.1.9

Find a basis for the eigenspace corresponding to each of the eigenvalues of \(A = \begin{pmatrix}3 & 0 \\ 2 & 1\end{pmatrix}\), $\lambda = 1, 3$

**Solution:** We need to find the set of vectors such that for each eigenvalue they meet

\[
A v = \lambda v \Rightarrow (A - \lambda I)v = 0
\]

$\lambda = 1$

\[
\begin{pmatrix}3 & 0 \\ 2 & 1\end{pmatrix} - \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}v = 0
\]

\[
\begin{pmatrix}2 & 0 \\ 2 & 0\end{pmatrix}v = 0
\]

The general solution of this homogeneous equation system is $v = \begin{pmatrix}0 \\ x_2\end{pmatrix}$. This is the eigenspace associated to the eigenvalue $\lambda = 1$ and one of its basis is \{(0,1)\}.

$\lambda = 3$
\[
\begin{pmatrix}
3 & 0 \\
2 & 1
\end{pmatrix} - 3
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0
\]
\[
\begin{pmatrix} 0 & 0 \\
2 & -2
\end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0
\]

The general solution of this homogeneous equation system is \( \mathbf{v} = \begin{pmatrix} x_2 \\ x_2 \end{pmatrix} \). This is the eigenspace associated to the eigenvalue \( \lambda = 3 \) and one of its basis is \( \{(1, 1)\} \).

**Lay, 5.1.17**

Find the eigenvalues of \( A = \begin{pmatrix} 0 & 0 & 0 \\
0 & 3 & 4 \\
0 & 0 & -2
\end{pmatrix} \).

**Solution:** We solve for \( \lambda \) the equation

\[
\begin{vmatrix}
-\lambda & 0 & 0 \\
0 & 3-\lambda & 4 \\
0 & 0 & -2-\lambda
\end{vmatrix} = -\lambda(3-\lambda)(-2-\lambda) = 0
\]

whose roots are \( \lambda = 0 \), \( \lambda = 3 \), and \( \lambda = -2 \).

**Lay, 5.1.19**

For \( A = \begin{pmatrix} 1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3
\end{pmatrix} \), find one eigenvalue, with no calculation. Justify your answer.

**Solution:** The determinant of \( A \) is zero because it has duplicated rows. On the other hand, the determinant is the product of the matrix eigenvalues, so at least one of the eigenvalues of \( A \) must be zero.

**Lay, 5.1.23**

Explain why a \( 2 \times 2 \) matrix can have at most two distinct eigenvalues. Explain why a \( n \times n \) matrix can have at most \( n \) distinct eigenvalues.

**Solution:** The eigenvalue problem

\[
|A - \lambda I| = 0
\]

implies finding the roots of a polynomial of degree \( n \) \( (|A - \lambda I|) \). Since a polynomial of degree \( n \) can have at most \( n \) distinct roots, then \( A \) can have at most \( n \) distinct eigenvalues.

**Lay, 5.1.25**

Let \( \lambda \) be an eigenvalue of an invertible matrix \( A \). Show that \( \lambda^{-1} \) is an eigenvalue of \( A^{-1} \). [Hint: suppose a non-zero \( x \) satisfies \( Ax = \lambda x \).]

**Solution:** Suppose

\[
Ax = \lambda x
\]

Let’s multiply on both sides by \( A^{-1} \)

\[
x = \lambda A^{-1}x
\]

\[
\lambda^{-1}x = A^{-1}x
\]
So \( \lambda^{-1} \) is an eigenvalue of \( A^{-1} \) and \( x \) is its associated eigenvector. It is noteworthy to see that \( x \) is an eigenvector of \( A \) and of \( A^{-1} \).

**Lay, 5.1.26**

Show that if \( A^2 \) is the zero matrix, then the only eigenvalue of \( A \) is 0.

**Solution:** Suppose \( \lambda \) is an eigenvalue of \( A \)

\[
Ax = \lambda x
\]

Let’s multiply on both sides by \( A \)

\[
A^2x = \lambda Ax
\]

\[
0 = \lambda(\lambda x)
\]

\[
0 = \lambda^2 x
\]

But \( x \) is non-zero (by the definition of eigenvector). Then, \( \lambda^2 = 0 \) and this implies that the only eigenvalue of \( A \) is 0.

**Lay, 5.1.27**

Show that \( \lambda \) is an eigenvalue of \( A \) if and only if \( \lambda \) is an eigenvalue of \( A^T \).

**Hint:** Find out how \( A - \lambda I \) and \( A^T - \lambda I \) are related.

**Solution:** Let us calculate the transpose of the matrix \( A - \lambda I \)

\[
(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I
\]

Now, by Theorem 6c of Section 2.2, \( (A - \lambda I)^T \) is not invertible if and only if \( A - \lambda I \) is not invertible. If \( \lambda \) is one of the eigenvalues, then \( A - \lambda I \) is not invertible. So \( (A - \lambda I)^T = A^T - \lambda I \) is not invertible either, and \( \lambda \) is one of the eigenvalues of \( A^T \).

**Lay, 5.2.1**

Find the characteristic equation and the real eigenvalues of the matrix

\[
A = \begin{pmatrix}
2 & 7 \\
7 & 2
\end{pmatrix}
\]

**Solution:** The characteristic equation is \( |A - \lambda I| = 0 \). In this particular case

\[
\begin{vmatrix}
2 - \lambda & 7 \\
7 & 2 - \lambda
\end{vmatrix}
= 0
\]

\[
(2 - \lambda)^2 - 49 = 0
\]

\[
4 + \lambda^2 - 4\lambda - 49 = 0
\]

\[
\lambda^2 - 4\lambda - 45 = 0
\]

\[
\lambda = \frac{4 \pm \sqrt{16 + 180}}{2} = \frac{4 \pm 14}{2} = \left\{ \begin{array}{c} 9 \\ -5 \end{array} \right. 
\]

The two real eigenvalues are \( \lambda = 9 \) and \( \lambda = -5 \).

**Lay, 5.2.9**

Find the characteristic equation and the real eigenvalues of the matrix

\[
A = \begin{pmatrix}
4 & 0 & -1 \\
0 & 4 & -1 \\
1 & 0 & 2
\end{pmatrix}
\]

**Solution:** The characteristic equation is \( |A - \lambda I| = 0 \). In this particular case
We calculate this determinant by expanding the factors and cofactors of the second column. Disregarding the cofactors multiplied by a zero value, we have

\[\begin{vmatrix} 4 - \lambda & 0 & -1 \\ 0 & 4 - \lambda & -1 \\ 1 & 0 & 2 - \lambda \end{vmatrix} = 0\]

Now, we factorize the term \((4 - \lambda)(2 - \lambda) + 1\)

\[(4 - \lambda)(2 - \lambda) + 1 = (8 + \lambda^2 - 6\lambda) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2\]

So the characteristic equation is

\[(4 - \lambda)(\lambda - 3)^2 = 0\]

whose solutions are \(\lambda = 4\) and \(\lambda = 3\) (with multiplicity 2).

Lay, 5.2.18

It can be shown that the algebraic multiplicity of an eigenvalue \(\lambda\) is always greater than or equal to the dimension of the eigenspace corresponding to \(\lambda\). Find \(h\) in the matrix below such that the eigenspace of \(\lambda = 4\) is two dimensional.

\[
A = \begin{pmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{pmatrix}
\]

Solution: Let’s calculate the eigenspace associated to the eigenvalue \(\lambda = 4\). For doing so we solve the homogenous equation \((A - \lambda I)x = 0\) making use of the augmented matrix below

\[
\begin{pmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & \frac{3}{h+3} & 0 & 0 \\ 0 & 0 & h + 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

Note that we have made the elements \(a_{13} = 0\) and \(a_{23} = 1\) because for doing that we would need to perform the row operations

\[
\begin{align*}
r_1 & \leftarrow r_1 - \frac{3}{h+3}r_2 \\
r_2 & \leftarrow \frac{1}{h+3}r_2
\end{align*}
\]

which are not allowed if \(h = -3\). If \(h \neq -3\), then the eigenspace is formed by all the vectors of the form \(\{x_1, 0, 0, 0\}\) whose dimension is 1. If \(h = -3\), then the eigenspace is formed by all the vectors of the form \(\{x_1, -\frac{3}{2}x_3, x_3, 0\}\) whose dimension is 2.

Lay, 5.2.19

Let \(A\) be an \(n \times n\) matrix, and suppose \(A\) has \(n\) real eigenvalues, \(\lambda_1, \lambda_2, ..., \lambda_n\), repeated according to multiplicities, so that
\[
\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\ldots(\lambda_n - \lambda)
\]

Explain why \(\det(A)\) is the product of the \(n\) eigenvalues of \(A\). (This result is true for any square matrix when complex eigenvalues are considered.)

**Solution:** Since the equation above is true for any value of \(\lambda\), we simply take \(\lambda = 0\) to obtain

\[
\det(A) = \lambda_1\lambda_2\ldots\lambda_n
\]

Lay, 5.2.20

Use a property of determinants to show that \(A\) and \(A^T\) have the same characteristic polynomial.

**Solution:** The characteristic polynomial of \(A\) is given by

\[
\det(A - \lambda I)
\]

For any matrix \(X\), we know that \(\det(X) = \det(X^T)\), then

\[
\det(A - \lambda I) = \det((A - \lambda I)^T) = \det(A^T - \lambda I) = \det(A^T - \lambda I)
\]

But this latter expression is the characteristic polynomial of \(A^T\).

Lay, 5.2.23

Show that if \(A = QR\) with \(Q\) invertible, then \(A\) is similar to \(A_1 = RQ\).

**Solution:** We remind that the matrices \(A\) and \(B\) are similar if there exists an invertible matrix \(P\) such that \(B = P^{-1}AP\) (with \(A, B, P \in M_{n \times n}\)). This means that we need to find an invertible matrix \(P\) such that

\[
A_1 = P^{-1}AP \\
RQ = P^{-1}QRP
\]

If we make \(P = Q\), since \(Q\) is invertible, we have \(P^{-1} = Q^{-1}\) and

\[
RQ = Q^{-1}QRP = RQ
\]

So, we have proven that \(A\) and \(A_1\) are similar.

Lay, 5.2.24

Show that if \(A\) and \(B\) are similar, then \(\det(A) = \det(B)\).

**Solution:** If \(A\) and \(B\) are similar, then there exists an invertible matrix \(P\) such that

\[
B = P^{-1}AP
\]

Applying the determinant on both sides we have

\[
\det(B) = \det(P^{-1}AP) = \det(P^{-1})\det(A)\det(P) = \frac{1}{\det(P)} \det(A)\det(P) = \det(A)
\]

Lay, 5.3.1

Let \(A = PDP^{-1}\) with \(P = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\) and \(D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\). Calculate \(A^4\).

**Solution:** If \(A = PDP^{-1}\), then

\[
A^4 = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})
\]

5
\[ A^4 = PD^4P^{-1} \]
\[ = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1^4 & 0 \\ 0 & 3^4 \end{pmatrix} \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix} \]
\[ = \begin{pmatrix} 32 & 160 \\ 480 & -239 \end{pmatrix} \]

**Lay, 5.3.23**

A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

**Solution:** According to Theorem 6.3.7, if the sum of the dimensions of the different eigenspaces is equal to the number of columns of A, then A is diagonalizable. This is the case of the matrix A of the problem for which 3 + 2 = 5, and consequently A is diagonalizable.

**Lay, 5.3.27**

Show that if A is both diagonalizable and invertible, then so is A⁻¹

**Solution:** If A is diagonalizable, then there exist an invertible matrix P and a diagonal matrix D such that

\[ A = PDP^{-1} \]

If A is invertible, then

\[ A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1} \]

So, D is also invertible and we see that A⁻¹ is also diagonalizable.

**Lay, 5.3.28**

Show that if A has n linearly independent eigenvectors, then so does Aᵀ. [Hint: Use the Diagonalization Theorem.]

**Solution:** The Diagonalization Theorem states that A is diagonalizable if and only if it has n linearly independent eigenvectors, and that in that case, A can be expressed as

\[ A = PDP^{-1}, \]

where the columns of P are the n linearly independent eigenvectors. Taking the transpose in both sides, we have

\[ A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D^T P^T \]

So, Aᵀ is also diagonalizable and, by the Diagonalization Theorem again, it must have n linearly independent eigenvectors.

**Lay, 5.3.29**

The diagonalization of a matrix is not unique. Given the following diagonalization of the matrix A

\[ \begin{pmatrix} 7 & 2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} \]
Now consider a new factorization of the form \( A = P_1 D_1 P_1^{-1} \) with \( D_1 = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \). Find the matrix \( P_1 \).

**Solution:** \( P_1 \) is simply the reorganization of the columns in \( P \) such that each eigenvector is in the same column as its corresponding eigenvalue in \( D_1 \).

\[
P_1 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}
\]

**Lay, 5.3.31**

Construct a \( 2 \times 2 \) matrix that is invertible but not diagonalizable.

**Solution:** The matrix \( A \) below is such a matrix

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\]

Its inverse is

\[
A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}
\]

But it is not diagonalizable. Let’s see why. Let’s calculate the eigenspace associated to \( \lambda = 1 \).

\[
(A - I)v = 0
\]

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}v = 0
\]

whose set of solutions is formed by all vectors of the form \( v = (x_1, 0) \) and its basis is \( \{(1, 0)\} \). Since the dimension of the eigenspace is 1 and there are 2 columns in \( A \), by Theorem 6.3.7, the matrix is not diagonalizable.

**Lay, 5.3.32**

Construct a \( 2 \times 2 \) matrix that is diagonalizable but not invertible.

**Solution:** Consider \( P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). Now let us construct the matrix

\[
A = PDP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

It is obviously diagonalizable by construction, but it is not invertible because one of its eigenvalues is 0, and \( D \) is not invertible.

**Lay, 5.4.1**

Let \( B = \{b_1, b_2, b_3\} \) and \( D = \{d_1, d_2\} \) be bases for vector spaces \( V \) and \( W \), respectively. Let \( T : V \to W \) be a linear transformation with the property that

\[
T(b_1) = 3d_1 - 5d_2
\]

\[
T(b_2) = -d_1 + 6d_2
\]

\[
T(b_3) = 4d_2
\]

Find the matrix of \( T \) relative to \( B \) and \( D \)

**Solution:** The matrix sought is

\[
\begin{pmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{pmatrix}
\]
$$M = ([T(b_1)]_B)_{3 	imes 3} = \begin{pmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \end{pmatrix}$$

We can apply it as

$$[T(x)]_B = M[x]_E$$

**Lay, 5.4.3**

Let $E = \{e_1, e_2, e_3\}$ be the standard basis for $\mathbb{R}^3$, let $B = \{b_1, b_2, b_3\}$ be a basis for a vector space $V$, and let $T : \mathbb{R}^3 \to V$ be a linear transformation with the property that $T(x_1, x_2, x_3) = (2x_3 - x_2)b_1 - (2x_2)b_2 + (x_1 + 3x_3)b_3$

a. Compute $T(e_1)$, $T(e_2)$ and $T(e_3)$.
b. Compute $[T(e_1)]_B$, $[T(e_2)]_B$ and $[T(e_3)]_B$.
c. Find the matrix for $T$ relative to $E$ and $B$

**Solution:**
a. Applying the transformation $T$ to the three standard vectors we get

\[
T(e_1) = T(1, 0, 0) = b_3 \\
T(e_2) = T(0, 1, 0) = -b_1 - 2b_2 \\
T(e_3) = T(0, 0, 1) = 2b_1 + 3b_3
\]
b. Let’s calculate now the coordinates of the different transformed vectors in $B$

\[
[T(e_1)]_B = (0, 0, 1) \\
[T(e_2)]_B = (-1, -2, 0) \\
[T(e_3)]_B = (2, 0, 3)
\]
c. The matrix sought is the one whose columns are the vectors in part b.

\[
M = \begin{pmatrix} 0 & 1 & 2 \\ 0 & -2 & 0 \\ 1 & 0 & 3 \end{pmatrix}
\]

**Lay, 5.4.5**

Let $T : P_2 \to P_3$ be the transformation that maps a polynomial $p(t)$ into the polynomial $(t + 3)p(t)$.
a. Find the image of $p(t) = 3 - 2t + t^2$
b. Show that $T$ is a linear transformation
c. Find the matrix for $T$ relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$. 

8
Solution:

a. $T(3 - 2t + t^2) = (t + 3)(3 - 2t + t^2) = 9 - 3t + t^2 + t^3$

b. We need to show that $T(p_1(t) + p_2(t)) = T(p_1(t)) + T(p_2(t))$ and $T(c(p(t)) = cT(p(t))$

• $T(p_1(t) + p_2(t)) = T(p_1(t)) + T(p_2(t))$

$$T(p_1(t) + p_2(t)) = (t + 3)(p_1(t) + p_2(t)) = (t + 3)p_1(t) + (t + 3)p_2(t) = T(p_1(t)) + T(p_2(t))$$

• $T(c(p(t)) = cT(p(t))$

$$T(c(p(t)) = (t + 3)(cp(t)) = c(t + 3)p(t) = c((t + 3)p(t)) = cT(p(t))$$

c. We need to calculate the transformation of each of the elements in the basis $\{1, t, t^2\}$

$$T(1) = (t + 3)1 = t + 3$$
$$T(t) = (t + 3)t = t^2 + 3t$$
$$T(t^2) = (t + 3)t^2 = t^3 + 3t^2$$

The matrix sought is

$$M = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

Lay, 5.4.13

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$ with $A = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$. Find a basis $B$ for $\mathbb{R}^2$ with the property $[T]_B$ is diagonal.

**Solution:** Let’s diagonalize $A$

$$A = PDP^{-1} = \begin{pmatrix} 0 & 1 \\ -3 & 4 \end{pmatrix}$$

If we construct the basis $B = \{(−0.3162, −0.9487), (−0.7071, −0.7071)\}$ is a basis in which the matrix of $T$ relative to $B$ is

$$[T]_B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

Lay, 5.4.18

Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x) = Ax$, where $A$ is a $3 \times 3$ matrix with eigenvalues 5, 5 and -2. Does there exist a basis $B$ for $\mathbb{R}^3$ such that the $B$-matrix of $T$ is a diagonal matrix? Discuss.
Solution: It depends on whether $A$ is diagonalizable or not. Since $A$ does not have all its eigenvalues distinct, the condition is (see Theorem 5.3.7) that the dimension of the eigenspace associated to eigenvalue 5 is 2, and that the dimension of the eigenspace associated to eigenvalue -2 is 1.

Lay, 5.4.22

If $A$ is diagonalizable and $B$ is similar to $A$, then $B$ is also diagonalizable.

Solution: If $A$ is diagonalizable, there exist an invertible matrix $P$ and a diagonal matrix $D$ such that

$$A = PDP^{-1}$$

If $B$ is similar to $A$, then there exists an invertible matrix $Q$ such that

$$B = QAQ^{-1}$$

Combining both results we have

$$B = Q(PDP^{-1})Q^{-1} = (QP)D(P^{-1}Q^{-1})$$

So $B$ is also diagonalizable since it can be expressed as

$$B = P'D(P')^{-1}$$

being $P' = QP$ an invertible matrix and $D$ a diagonal matrix.

Lay, 5.4.23

If $B = P^{-1}AP$ and $x$ is an eigenvector of $A$ corresponding to an eigenvalue $\lambda$, then $P^{-1}x$ is an eigenvector of $B$ corresponding also to an eigenvalue $\lambda$.

Solution: Let’s check whether the statement proposed by the problem is true or not. If it is true, it means that

$$B(P^{-1}x) = \lambda(P^{-1}x)$$

According to the problem we have that $B = P^{-1}AP$, so

$$B(P^{-1}x) = (P^{-1}AP)(P^{-1}x) = P^{-1}Ax$$

But by hypothesis $x$ is an eigenvector of $A$ corresponding to an eigenvalue $\lambda$, that is $Ax = \lambda x$. Consequently,

$$P^{-1}Ax = P^{-1}(\lambda x) = \lambda(P^{-1}x)$$

Finally, we have proven that, as stated by the problem,

$$B(P^{-1}x) = \lambda(P^{-1}x)$$

Lay, 5.4.25

The trace of a square matrix $A$ is the sum of the diagonal entries in $A$ and is denoted as $\text{tr}\{A\}$. It can be verified that $\text{tr}\{FG\} = \text{tr}\{GF\}$ for any two $n \times n$ matrices $F$ and $G$. Show that if $A$ and $B$ are similar, then $\text{tr}\{A\} = \text{tr}\{B\}$.

Solution: If $A$ and $B$ are similar, then there exists an invertible matrix $P$ such that

$$B = P'\text{tr}\{A\} \text{tr}\{B\}$$
\[ B = PAP^{-1} \]

Taking the trace of both sides

\[
\text{tr}\{B\} = \text{tr}\{PAP^{-1}\} = \text{tr}\{P(AP^{-1})\} = \text{tr}\{(AP^{-1})P\} = \text{tr}\{A(P^{-1}P)\} = \text{tr}\{A\}
\]

**Lay, 5.4.26**

It can be shown that the trace of a matrix equals the sum of its eigenvalues. Verify this statement for the case when \(A\) is diagonalizable.

**Solution:** If \(A\) is diagonalizable, then \(A = PDP^{-1}\), that is \(D\) is similar to \(A\). Then, by Exercise 5.4.25

\[
\text{tr}\{A\} = \text{tr}\{D\} = \sum_{i=1}^{n} \lambda_i
\]

**Lay, 5.4.27**

Let \(V\) be \(\mathbb{R}^n\) with a basis \(B = \{b_1, b_2, \ldots, b_n\}\); let \(W\) be \(\mathbb{R}^n\) with the standard basis, denoted here by \(E\); and consider the identity transformation \(I : V \to W\), \(I(x) = x\). Find the matrix for \(I\) relative to \(B\) and \(E\). What was this matrix called in the context of coordinate systems (Section 4.4)?

**Solution:** The transformation matrix is given by

\[
M = \begin{pmatrix}
[I(b_1)]_E & [I(b_2)]_E & \cdots & [I(b_n)]_E
\end{pmatrix} = \begin{pmatrix}
b_1 & b_2 & \cdots & b_n
\end{pmatrix}
\]

This was the change of coordinates matrix in Section 4.4, denoted as \(P_{E \leftarrow B}\).

**Lay, 5.5.1**

Let the matrix \(A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}\) act on \(\mathbb{C}^2\). Find the eigenvalues and a basis for each of the eigenspace in \(\mathbb{C}^2\).

**Solution:** The eigenvalues are the solutions of the characteristic equation

\[
|A - \lambda I| = 0
\]

\[
\begin{vmatrix}
1 - \lambda & -2 \\
1 & 3 - \lambda
\end{vmatrix} = (1 - \lambda)(3 - \lambda) + 2 = (\lambda - (2 + i))(\lambda - (2 - i)) = 0
\]

The two eigenvalues are complex. Let’s find now a basis for each one of the eigenspaces.

\[
\lambda = 2 + i
\]

We need to solve the homogeneous equation system \((A - \lambda I)v = 0\)

\[
\begin{pmatrix}
1 - (2 + i) & -2 \\
1 & 3 - (2 + i)
\end{pmatrix}v = 0
\]

We use the augmented matrix below
All vectors in this eigenspace are of the form $v = ((-1+i)x_2, x_2) \in \mathbb{R}$. One of its bases is $\{(1-i, 1)\}$

$$\lambda = 2 - i$$

We need to solve the homogeneous equation system $(A - \lambda I)v = 0$

$$\begin{pmatrix} 1 - (2 - i) & -2 \\ 1 & 3 - (2 - i) \end{pmatrix}v = 0$$

We use the augmented matrix below

$$\begin{pmatrix} -1 + i & -2 & 0 \\ 1 & 1 + i & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 + i & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

All vectors in this eigenspace are of the form $v = ((-1 - i)x_2, x_2) \in \mathbb{R}$. One of its bases is $\{(-1 - i, 1)\}$

In fact this is a general result, if $\lambda$ and $\lambda^*$ are two complex conjugate eigenvalues, then their corresponding bases are also related by a complex conjugate operation.

**Lay, 5.5.7**

Let the matrix $A = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$. Consider the transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as $T(x) = Ax$. $T$ is the composition of a scaling and a rotation. Give the scaling factor and the rotation angle.

**Solution:** The eigenvalues of $A$ are

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} \sqrt{3} - \lambda & -1 \\ 1 & \sqrt{3} - \lambda \end{vmatrix} = (\sqrt{3} - \lambda)^2 + 1 = 0$$

$$\lambda = \sqrt{3} \pm i = 2e^{\pm 30^\circ}$$

The scaling factor is 2 and the rotation angle 30° or -30° (in fact looking only at the eigenvalues we cannot determine the sign of the rotation). However, we note that

$$A = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{pmatrix}$$

So, the rotation angle is 30°.

**Lay, 5.5.13**

Let the matrix $A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$. Find an invertible matrix $P$ and a matrix $C$ of the form $\begin{pmatrix} b & -a \\ a & b \end{pmatrix}$ such that $A = PCP^{-1}$.

**Solution:** As in Exercise 5.5.1, the eigenvalues of $A$ are $\lambda = 2 \pm i$. A basis of the eigenvalue $\lambda = 2 - i$ is $\{(-1 - i, 1)\}$. According to Theorem 5.5.9, $\lambda = a - bi$ and $v$ is a basis of its eigenspace, we find the $P$ and $C$ matrices as
\[ P = (\text{Re}\{v\} \quad \text{Im}\{v\}) \]
\[ C = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \]

In this case,
\[ P = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \]
\[ C = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \]

It can be easily verified that
\[ A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}^{-1} \]

**Lay, 5.5.23**

Let \( A \) be an \( n \times n \) real matrix with the property that \( A^T = A \), let \( x \) be any vector in \( \mathbb{C}^n \), and let \( q = (x^*)^T A x \). Show that \( q \) is a real number.

**Solution:** We need to show that \( q^* = q \).
\[
q^* = ((x^*)^T A x)^* = (AB)^* = A^* B^*; (x^*)^T = (x^T)^* \]
\[
= x^T A^* x^* \quad \text{by hypothesis} \quad A \text{ is real} \]
\[
= x^T A x \quad \text{by hypothesis} \quad A^T = A \]
\[
= q \]

**Lay, 5.5.24**

Let \( A \) be an \( n \times n \) real matrix with the property that \( A^T = A \). Show that if \( Ax = \lambda x \) for some nonzero vector \( x \in \mathbb{C}^n \), then, in fact, \( \lambda \) is real and the real part of \( x \) is an eigenvector of \( A \). [Hint: Compute \((x^*)^T A x\) and use Exercise 5.5.23. Also, examine the real part of \( Ax \).]

**Solution:** Let us calculate \( q = (x^*)^T A x \)
\[
q = (x^*)^T A x = (x^*)^T (Ax) = (x^T)^T (\lambda x) = \lambda \|x\|^2 \]

Since \( \|x\|^2 \) is a real number and \( q \) is a real number, then \( \lambda \) is a real number.

Let us calculate the real part on both sides of the equation \( Ax = \lambda x \)
\[
\text{Real}\{Ax\} = \text{Real}\{\lambda x\} \quad [\lambda \text{ and } \lambda \text{ are real}] \]
\[
A\text{Real}\{x\} = \lambda \text{Real}\{x\} \]

So \( \text{Real}\{x\} \) is an eigenvector of \( A \).

**Lay, 5.5.25**

Let \( A \) be a real \( n \times n \) matrix, and let \( x \in \mathbb{C}^n \). Show that \( \text{Real}\{Ax\} = A\text{Real}\{x\} \) and \( \text{Imag}\{Ax\} = A\text{Imag}\{x\} \).

**Solution:** Consider
\[
x = \text{Real}\{x\} + i\text{Imag}\{x\} \]
Multiplying on both sides by $A$ on the left:

$$Ax = A(\text{Real}\{x\} + i\text{Imag}\{x\})$$

$$= A\text{Real}\{x\} + iA\text{Imag}\{x\}$$

Now simply by taking the real and imaginary parts of $Ax$ and taking into account that $A$ is a real matrix, we get the properties proposed:

$$\text{Real}\{Ax\} = A\text{Real}\{x\}$$

$$\text{Imag}\{Ax\} = A\text{Imag}\{x\}$$